Master CMB, 1st year
Differential systems, semester 1

TP - Linear ODEs

During the session, we shall need the following libraries: `numpy` (the fundamental package for scientific computing in Python), `scipy` (in particular `scipy.integrate` which contains a solver of differential equations), `matplotlib` (for the representation of the solutions). You can import these libraries as follows:

```
import numpy as np
import scipy.integrate as spi
import matplotlib.pyplot as plt
```

Exercise 1 (Pharmaco-kinetics)
We consider the following single compartment model

$$\begin{cases} Q_a'(t) = -k_a Q_a(t) \\ Q'(t) = -k_e Q(t) + k_a Q_a(t) \end{cases} \tag{1}$$

with $k_a > 0,\ k_e \geq 0$, and the initial condition

$$Q_a(0) = D = 10, \quad Q(0) = 0.$$

1. Fix $k_a = 2$, $k_e = 1$. Determine the equilibrium point(s) y^* of the system.
   For that purpose, you can define the matrix A associated to the differential system as follows

   ```
   # matrix A
   def Mat(ka,ke):
           A = np.array([[-ka, 0.],[ka, -ke]])
           return A
   ```

   and then use the function `np.linalg.solve` to determine y^*.

2. In order to compute an approximate solution to our differential system, one can use the command `odeint` from the library `scipy.integrate`.

   (a) Here is how to use this function to solve the equation satisfied by Q_a.

   ```
   # parameters
   ka = 2.
   D = 10.

   # time discretization
   t0 = 0.  # initial time
   T = 10.  # final time
   dt = 0.05  # time step
   tt = np.arange(t0,T,dt)

   # definition of the right-hand side of the ODE Qa'=fa(y,t,args)
   def fa(y,t,ka):
           z = -ka*y
           return z

   y0 = D # initial condition

   Qa = spi.odeint(fa,y0,tt,args=(ka,))
   ```

Represent the time evolution of the computed numerical solution using `plt.plot`.

(b) Define the function `pharmaco1`

```
def pharmaco1(Y,t,ka,ke):
        A = Mat(ka,ke)
        Z = np.dot(A,Y)
        return Z
```

and solve directly the full system (1) using again `spi.odeint` applied to the function `pharmaco1`. Represent Q and Q_a on the same figure.

3. Compare the solutions Q for different values of the parameter k_e, for instance: $k_e = 0,\ 1,\ 2,\ 5$. Comment the results.

4. (*) We consider a three compartments model describing the time evolution of the concentrations $c_1,\ c_2,\ c_3$ in each compartment:

$$C'(t) = AC(t) + b(t) \tag{2}$$

with

$$A = \begin{pmatrix} -(k+ke) & 0 & 0 \\ k & -k & 0 \\ 0 & k & -k \end{pmatrix}, \quad b(t) = \begin{pmatrix} k_a D e^{-kat} \\ 0 \\ 0 \end{pmatrix}.$$

For the two sets of parameters: $(k_e, k) = (0, 3)$ and $(3, 3)$, compute the eigenvalues and eigenvectors of the matrix A by using the function `np.linalg.eig`. Compute and represent the numerical solution of the differential system (2) obtained from the initial condition: $c_1(0) = c_2(0) = c_3(0) = 0$ and for instance $k_a = 2$.

Exercise 2 (RLC circuit)
We consider the second order differential equation modeling the evolution of the voltage across a capacitor in a series RLC circuit:

$$y''(t) + \frac{R}{L}y'(t) + \frac{1}{LC}y(t) = E(t)$$

which rewrites as

$$Y'(t) = AY(t) + b(t) \quad \text{with} \quad Y = \begin{pmatrix} y \\ y' \end{pmatrix}. \tag{3}$$

1. Set $E(t) = 0,\ R = 0.3,\ L = 0.8,\ C = 1$. Represent the solution curves and the trajectories obtained from the initial data

$$(y(0), y'(0)) = (1, 0) \quad \text{and} \quad (y(0), y'(0)) = (-1, 0).$$

2. On the same graph as the trajectories, represent the phase portrait using the command `plt.quiver` as follows:

```
xmin = -2
xmax = 2
ymin = xmin
ymax = xmax

xx=np.linspace(xmin,xmax,20)
yy=xx
```

```
XX,YY=np.meshgrid(xx,yy)
FX=0*XX
FY=0*YY
n,p=np.shape(XX)
for i in range(n):
        for j in range(p):
                Z = RLC([XX[i,j],YY[i,j]],0,R,L,C)
                FX[i,j] = Z[0]
                FY[i,j] = Z[1]

plt.quiver(XX,YY,FX,FY)
```

where `RLC` is the function associated to the right-hand side of (3).

3. Compare the solutions obtained for different values of the parameter R, for instance $R = 0,\ 0.3,\ 5$.
   Introduce the energy of the system at time t defined as

$$\mathcal{E}(t) = \frac{1}{2}\left[(y'(t))^2 + \frac{(y(t))^2}{LC}\right]$$

   and represent its time evolution in the three cases $R = 0,\ 0.3,\ 5$. Comment the result.

4. (*) Consider $R = 0$, $E(t) = \cos(\Omega t)$. Compare the different cases $\Omega = 0.6,\ \omega_0,\ 2$, what do you observe ?