**Master CMB, 1st year**
**Differential systems, semester 1**

**TP - Nonlinear ODEs**

**Exercise 1** (The van der Pol oscillator)
We consider the nonlinear differential equation

$$x''(t) - \mu(1 - x(t)^2)x'(t) + x(t) = 0$$

which can be rewritten under the following form

$$\begin{cases} x'(t) = \mu \left( x(t) - \dfrac{x(t)^3}{3} \right) + y(t) & \text{(1a)} \\ y'(t) = -x(t) & \text{(1b)} \end{cases}$$

1. Let us fix $\mu = 1$. Represent on the same graph the solutions curves (resp. the trajectories) for the following initial data: $(0, 1)$, $(3, 3)$, $(-3, -3)$. What do you observe ?

2. Determine the equilibrium point(s) of the system. According to the value of $\mu$, illustrate the nature (local stability or instability) of the equilibrium point(s) by representing the numerical solutions associated to the linearized system(s) around the equilibrium point(s).

3. We fix again $\mu = 1$. Represent in the phase plane the nullclines associated to system (1). What is coarsely the behavior the solutions in each region of the plane ? Complete the phase portrait with the vector field and the limit cycle.

4. Compare the case $\mu = 1$ with the case $\mu = 20$. Interpret the behavior of the solutions in that latter case.

**Exercise 2** (The FitzHugh Nagumo system)
We consider the following extension of the previous van der Pol system (1)

$$\begin{cases} x'(t) = \mu \left( x(t) - \dfrac{x(t)^3}{3} \right) + y(t) + I & \text{(2a)} \\ y'(t) = -x(t) + a - by(t) & \text{(2b)} \end{cases}$$

1. We set $\mu = 6$, $a = 1$, $b = 2$, $I = -\frac{a}{b}$.

   (a) Determine the nullclines and the equilibrium point(s) of the system. Represent them in the phase plane with the vector field.

   (b) Determine the nature of the equilibrium point(s) and illustrate it graphically.

2. We set $\mu = 6$, $b = I = 0$. We now study the behavior of the solutions for different values of $a$.

   (a) Determine the nullclines and the equilibrium point(s). What can you say about the nature of the equilibrium point(s) according to the value of the parameter $a$.

   (b) Illustrate the results numerically.