Université d'Aix-Marseille 2019-2020 | Master CMB-Bio

Continuous dynamical systems and modeling

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Exercise 1. We consider a specie whose population (i.e. the number of individuals) has doubled in 100 years and tripled in 200 years. Show that this population cannot follow the Mathusian dynamics.

Correction. Let us denote $N(t)$ the size of the population at time t. We know that

$$N(100) = 2N(0), \tag{1}$$

$$N(200) = 3N(0). \tag{2}$$

If N follows the Malthusian dynamics, then there exists $a \in \mathbb{R}$ such that

$$N'(t) = aN(t) \quad \forall\, t \in \mathbb{R},$$

and thus

$$N(t) = N(0)e^{at} \quad \forall\, t \in \mathbb{R},$$

where $N(0) > 0$. On the one hand, we have from (1):

$$N(100) = 2N(0) = N(0)e^{100a} \qquad \Longrightarrow \qquad a = \frac{\ln(2)}{100}.$$

On the other hand, due to (2), it also holds:

$$N(200) = 3N(0) = N(0)e^{200a} \qquad \Longrightarrow \qquad a = \frac{\ln(3)}{200}.$$

Since $\dfrac{\ln(2)}{100} \neq \dfrac{\ln(3)}{200}$, we arrive at a contradiction, which means that N does not follow a Malthusian dynamics.

Alternative method. Write that

$$\begin{aligned} N(200) &= N(0)e^{200a} \\ &= N(0)e^{100a} \times e^{100a} \\ &= \frac{1}{N(0)}\big(N(0)e^{100a}\big)^2 \\ &= \frac{1}{N(0)}\big(N(100)\big)^2 \\ &= \frac{1}{N(0)}\big(2N(0)\big)^2 \quad \text{using (1)} \\ &= 4N(0). \end{aligned}$$

By (2): $N(200) = 3N(0) \neq 4N(0)$ (since $N(0) > 0$), which leads again to a contradiction.

Exercise 2. *Carbon-14 dating*
The carbon contained in living matter consists essentially in the stable isotope C^{12} (6 neutrons and 6 protons), but a tiny part is represented the radio-active isotope C^{14} (8 neutrons and 6 protons). This radio-active component comes from the cosmic radiation in the upper atmosphere. Thanks to a complex exchange process, the living matter maintains a constant portion of C^{14} in its total mass of carbon.
After death, the exchanges cease and the quantity of radio-active carbon decreases: it looses 1/8000 of its mass every year. This process enables scientists to date the death of the individuals from the analysis of the bones.

(1) Determine the differential equation satisfied by the mass of carbon 14 contained in a "dead" bone.

(2) Represent the shape of the solution curves for different initial conditions.

(3) Applications:

   (a) Skeletons from Neanderthal humans have been found in Palestine. Analysis shows that the portion of C^{14} is about 6.24% of the mass of C^{14} contained in a living body. Estimate the age of the skeleton.

   (b) Conversely, one estimates that Cro-Magnon man was living in France in the period between 30000 and 20000 BC. Determine an interval for the ratio between the mass of C^{14} in the skeleton and the one in a living body.

Correction.

(1) Let us denote $C(t)$ the mass of C^{14} at time t. We assume that $t_0 = 0$ is the time of the death of the individual. Therefore, $C(0) > 0$ corresponds to the mass of C^{14} contained in a living body. We know that there is a loss of 1/8000 of the mass every year, which means that

$$\frac{C(t+\Delta t) - C(t)}{\Delta t} = -\frac{1}{8000} C(t)$$

where

$$\Delta t = 1 \text{ year}$$

is supposed to be very small compared to the expected age of the skeletons. Hence

$$\frac{C(t+\Delta t) - C(t)}{\Delta t} \approx \frac{d}{dt} C(t)$$

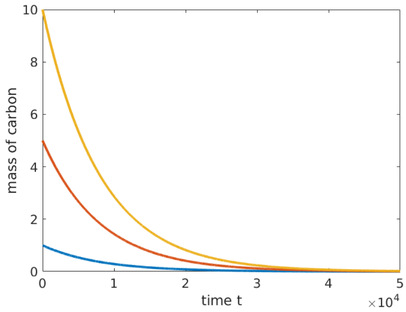
and we arrive at the differential equation

$$\boxed{C'(t) = -\frac{1}{8000} C(t)} \qquad (3)$$

(2) C satisfies the Malthusian dynamics:

$$C(t) = C(0) e^{-\frac{t}{8000}} \quad \forall t \geq 0.$$

Here is a representation of the solutions associated to different initial conditions.



It is important to note that the solution curves do not intersect.

(3) Applications

(a) We denote t^* the age of the skeleton of the Neanderthal man. We have

$$C(t^*) = C(0)e^{-\frac{t^*}{8000}} = \frac{6.24}{100}C(0).$$

Therefore

$$e^{-\frac{t^*}{8000}} = \frac{6.24}{100} \quad \Longrightarrow \quad \boxed{t^* = -8000 \times \ln\left(\frac{6.24}{100}\right)} \approx 22194 \text{ years}$$

(b) Let us set $t_- = 30000 + 2000 = 32000$ and $t_+ = 22000$. The expected mass of C^{14} at times $t_\pm$ is

$$C_- = C(t_-) = C(0)e^{\frac{-t_-}{8000}} = C(0)e^{-\frac{32000}{8000}} = C(0)e^{-4} \approx \frac{1.83}{100}C(0),$$

$$C_+ = C(t_+) = C(0)e^{\frac{-t_+}{8000}} = C(0)e^{-\frac{22000}{8000}} = C(0)e^{-\frac{11}{4}} \approx \frac{6.39}{100}C(0).$$

The interval is then $\left[\frac{1.83}{100}, \frac{6.39}{100}\right]$.