Aix-Marseille Université Master CMB-Bio
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Continuous dynamical systems and modeling
Linear ODEs

Exercise 1 Solve the following one-dimensional differential equations.

1. $(E_1)\ :\ 2y'(t) + y(t) = 5;$

2. $(E_2)\ :\ y'(t) - 2y(t) = e^{4t}.$

Correction.

1. The homogeneous equation reads

$$2y'(t) + y(t) = 0$$

or written differently

$$y'(t) = -\frac{1}{2}y(t).$$

The general solution of the homogeneous equation, y_H, then writes

$$y_H(t) = Ke^{-\frac{t}{2}} \quad \text{for some } K \in \mathbb{R}.$$

Now, following the method of the variation of the constant, we look for a particular solution y_P of the original (non-homogeneous) equation (E_1) under the form

$$y_P(t) = K(t)e^{-\frac{t}{2}}$$

where K is a function to determine. If y_P is a solution of (E_1) then

$$2y_P'(t) + y_P(t) = 5,$$

and thus

$$2\left[K'(t) - \frac{K(t)}{2}\right]e^{-\frac{t}{2}} + K(t)e^{-\frac{t}{2}} = 5.$$

Simplifying some terms in the left-hand side, we obtain

$$2K'(t)e^{-\frac{t}{2}} = 5 \quad \Longrightarrow \quad K'(t) = \frac{5}{2}e^{\frac{t}{2}} \quad \Longrightarrow \quad K(t) = 5e^{\frac{t}{2}} + K_0,\ K_0 \in \mathbb{R}.$$

Therefore, a particular solution of (E_1) is given by

$$y_P(t) = K(t)e^{-\frac{t}{2}} = K_0e^{-\frac{t}{2}} + 5,\ K_0 \in \mathbb{R}.$$

Applying the superposition principle, the general solution y of (E_1) writes

$$y(t) = y_H(t) + y_P(t) = Ce^{-\frac{t}{2}} + 5 \quad \text{for some } C \in \mathbb{R}.$$

Remark: The constant C is determined by the initial condition. If, for instance, we add the initial condition $y(0) = y_0$ then

$$C + 5 = y_0 \quad \text{and} \quad y(t) = (y_0 - 5)e^{-\frac{t}{2}} + 5.$$

2. The homogeneous equation reads

$$y'(t) - 2y(t) = 0$$

or written differently

$$y'(t) = 2y(t).$$

The general solution of the homogeneous equation, y_H, then writes

$$y_H(t) = Ke^{2t} \quad \text{for some } K \in \mathbb{R}.$$

Now, following the method of the variation of the constant, we look for a particular solution y_P of the original (non-homogeneous) equation (E_2) under the form

$$y_P(t) = K(t)e^{2t}$$

where K is a function to determine. If y_P is a solution of (E_1) then

$$y_P'(t) - 2y_P(t) = e^{4t},$$

and thus

$$\left[K'(t) + 2K(t)\right] e^{2t} - 2K(t)e^{2t} = e^{4t}.$$

Simplifying some terms in the left-hand side, we obtain

$$K'(t)e^{2t} = e^{4t} \quad \Longrightarrow \quad K'(t) = e^{2t} \quad \Longrightarrow \quad K(t) = \frac{1}{2}e^{2t} + K_0,\ K_0 \in \mathbb{R}.$$

Therefore, a particular solution of (E_1) is given by

$$y_P(t) = K(t)e^{2t} = K_0e^{2t} + \frac{1}{2}e^{4t},\ K_0 \in \mathbb{R}.$$

Applying the superposition principle, the general solution y of (E_2) writes

$$y(t) = y_H(t) + y_P(t) = Ce^{2t} + \frac{1}{2}e^{4t} \quad \text{for some } C \in \mathbb{R}.$$

Exercise 2 [Pharmacokinetics - 1] We consider a single compartment model where the drug is orally administered. In this model, the drug is first absorbed by the stomach. Then, once the drug is present in the blood, it is eliminated by the organism. Let us denote $Q_a(t)$ the quantity of drug in the stomach at time t, $Q(t)$ the quantity of drug in the blood. We describe the dynamics of the whole process by the following differential system

$$\begin{cases} Q_a'(t) = -k_a Q_a(t) \\ Q'(t) = -k_e Q(t) + k_a Q_a(t) \end{cases}$$

where we assume that $k_a, k_e > 0$, $k_a \neq k_e$.

1. Determine the equilibrium points of the system.

2. Give the general form of the solutions.

3. For an oral administration, it is natural to set

$$Q_a(0) = D, \quad Q(0) = 0.$$

Determine the solution associated to this initial condition. What is its behaviour as $t \to +\infty$?

Correction.

1. The differential system can be recast as

$$Y'(t) = AY(t) \quad \text{with} \quad Y = \begin{pmatrix} Q_a \\ Q \end{pmatrix}, \quad A = \begin{pmatrix} -k_a & 0 \\ k_a & -k_e \end{pmatrix}$$

The equilibrium points $Y^* = \begin{pmatrix} Q_a^* \\ Q^* \end{pmatrix}$ of the differential system satisfy

$$AY^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that is

$$\begin{cases} -k_a Q_a^* = 0 \\ -k_e Q^* + k_a Q_a^* = 0 \end{cases} \quad \Longrightarrow \quad Q_a^* = 0,\ Q^* = 0.$$

Hence, there is a unique equilibrium point: the origin $(0, 0)$.

2. Since the evolution of the quantity Q_a does not depend on Q, we can first solve the differential equation satisfied by Q_a and then replace Q_a in the second equation to find Q. The general solution of the differential equation

$$Q_a'(t) = -k_a Q_a(t)$$

is given by

$$Q_a(t) = K_1 e^{-k_a t}, \quad K \in \mathbb{R}.$$

Therefore Q satisfies the equation

$$Q'(t) = -k_e Q(t) + k_a Q_a(t) = -k_e Q(t) + k_a K_1 e^{-k_a t}.$$

The associated homogeneous equation reads

$$Q'(t) = -k_e Q(t)$$

whose solutions, Q_H, take the form

$$Q_H(t) = \tilde{K} e^{-k_e t}, \quad \tilde{K} \in \mathbb{R}.$$

Applying the method of the variation of the constant, we look for a particular solution Q_P of the complete equation under the form

$$Q_P(t) = \tilde{K}(t) e^{-k_e t}.$$

Plugging this expression into the equation, we find that the function $\tilde{K}$ has to satisfy

$$\tilde{K}'(t) e^{-k_e t} = k_a K_1 e^{-k_a t}$$

that is

$$\tilde{K}'(t) = k_a K_1 e^{(k_e - k_a)t}.$$

Therefore, since we have assumed $k_e \neq k_a$, we get

$$\tilde{K}(t) = -\frac{k_a}{k_a - k_e} K_1 e^{-(k_a - k_e)t} + \tilde{K}_0, \quad \tilde{K}_0 \in \mathbb{R}.$$

Finally, the general solution is given by

$$\begin{cases} Q_a(t) = K_1 e^{-k_a t} \\ Q(t) = -\dfrac{k_a}{k_a - k_e} K_1 e^{-k_a t} + K_2 e^{-k_e t} \end{cases} \qquad K_1,\ K_2 \in \mathbb{R}.$$

Remark: If we had assumed that $k_a = k_e$, then we would have

$$Q'(t) = -k_e(Q(t) - Q_a(t)) = -k_e(Q(t) - K_1 e^{-k_e t}).$$

Following the same lines for the solving of this differential equation, we would have obtained the same expression for the solution Q_H of the homogeneous system but a different formula for the particular solution Q_P. Indeed, in the case $k_e = k_a$, we get

$$\tilde{K}'(t) = k_e K_1$$

and then

$$\tilde{K}(t) = k_e K_1 t + \tilde{K}_0 \quad \Longrightarrow \quad Q_P(t) = \tilde{K}(t) e^{-k_e t} = (k_e K_1 t + \tilde{K}_0) e^{-k_e t}$$

Finally, in the case $k_e = k_a$, the general solution is given by

$$\begin{cases} Q_a(t) = K_1 e^{-k_e t} \\ Q(t) = \big(K_1 k_e t + K_2\big) e^{-k_e t} \end{cases} \qquad K_1,\ K_2 \in \mathbb{R}.$$

3. If we impose the initial condition

$$Q_a(0) = D, \quad Q(0) = 0,$$

then we can fix the constants $K_{1,2}$. They satisfy in the case $k_e \neq k_a$

$$K_1 = D, \qquad -\frac{k_a}{k_a - k_e} K_1 + K_2 = 0$$

and thus

$$K_1 = D, \quad K_2 = \frac{k_a}{k_a - k_e} D.$$

The solution then writes

$$\begin{cases} Q_a(t) = De^{-k_a t} \\ Q(t) = \dfrac{k_a}{k_a - k_e} D \left(e^{-k_e t} - e^{-k_a t}\right) \end{cases}$$

We can check that $Q_a(t),\ Q(t) > 0$ for all times $t \geq 0$ and as $t \to +\infty$, since $k_a,\ k_e > 0$ we obtain

$$\lim_{t \to +\infty} Q_a(t) = 0, \qquad \lim_{t \to +\infty} Q(t) = 0.$$

Remark: In the case $k_a = k_e$, we would have obtained

$$K_1 = D, \quad K_2 = 0 \quad \text{and thus} \quad \begin{cases} Q_a(t) = De^{-k_e t} \\ Q(t) = Dk_e t\, e^{-k_e t} \end{cases}$$

As $t \to +\infty$, we observe that we still have

$$\lim_{t \to +\infty} Q_a(t) = 0, \qquad \lim_{t \to +\infty} Q(t) = 0.$$

Exercise 3 Determine the general solution of the differential system

$$Y'(t) = AY(t)$$

in the two cases

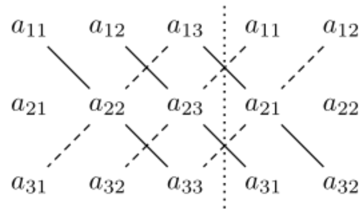
$$A = A_1 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad A = A_2 = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

Recalls on the calculation of the determinant of a matrix.

- If A is a 2×2 matrix we recall that the determinant of A is given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

- If A is a 3×3 matrix, its determinant can be calculated
  - either by the Sarrus rule: the sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements, when the copies of the first two columns of the matrix are written beside it as in the illustration.

    The final calculation gives

    $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

    $$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

    This method cannot be extended to higher dimension (i.e. for matrix $A \in M_n(\mathbb{R})$ with $n > 3$).
  - or by a Laplace expansion (which can be generalized to higher dimension): choose one row (or one column) and expand the determinant by this row (or this column). For instance, the expansion by the row i writes

    $$\det(A) = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} \det(M_{ij})$$

    where $M_{ij} \in M_2(\mathbb{R})$ is the matrix extracted from A where we have removed the row i and the column j. Despite this abstract formula, this method is convenient in practice for 3×3 matrix, specially if there are some 0 in the matrix (and it applies in any dimension). More precisely, if we expand the determinant of A by the 2nd column (for instance) we get

    $$\det(A) = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

    One can check that we recover in the end the same result as with the Sarrus rule. See details here:
    https://www.wikihow.com/Find-the-Determinant-of-a-3X3-Matrix

Correction.

1. Let us calculate the characteristic polynomial of the matrix A_1:

$$\chi_{A_1}(X) := \begin{vmatrix} -X & 1 & 1 \\ -1 & 2-X & 1 \\ 1 & 0 & 1-X \end{vmatrix}.$$

To calculate the determinant, one can expand for instance by the 3rd row:

$$\begin{aligned}
\chi_{A_1}(X) &= 1 \times \begin{vmatrix} 1 & 1 \\ 2-X & 1 \end{vmatrix} - 0 \times \begin{vmatrix} -X & 1 \\ -1 & 1 \end{vmatrix} + (1-X) \times \begin{vmatrix} -X & 1 \\ -1 & 2-X \end{vmatrix} \\
&= 1 - (2-X) + (1-X)\Big(-X(2-X) + 1 \Big) \\
&= (X-1)\Big(-1 - X(2-X) + 1 \Big) \\
&= -X(X-1)(X-2).
\end{aligned}$$

As a consequence, the matrix A_1 admits three distinct eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

Let us calculate eigenvectors V_i associated to the eigenvalues λ_i:

- an eigenvector $V_1 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ associated to λ_1 satisfies the equation

  $$A_1 V_1 = \lambda_1 V_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

  equality which can be rewritten as the following linear system

  $$\begin{cases} v_2 + v_3 = 0 \\ -v_1 + 2v_2 + v_3 = 0 \\ v_1 + v_3 = 0 \end{cases}$$

  Observing that the second equation is redundant with the two others ($L_2 = 2L_1 - L_3$), we get

  $$\begin{cases} v_2 + v_3 = 0 \\ v_1 + v_3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} v_3 = -v_1 \\ v_2 = -v_3 = v_1 \end{cases}$$

  Therefore $V_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_1 = 0$.

- an eigenvector $V_2 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ associated to λ_2 satisfies the equation

  $$A_1 V_2 = \lambda_2 V_2 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

  equality which can be rewritten as the following linear system

  $$\begin{cases} v_2 + v_3 = v_1 \\ -v_1 + 2v_2 + v_3 = v_2 \\ v_1 + v_3 = v_3 \end{cases}$$

Observing that the second equation is redundant with the two others, we get

$$\begin{cases} v_2 + v_3 = v_1 \\ v_1 + v_3 = v_3 \end{cases} \quad \text{and then} \quad \begin{cases} v_1 = 0 \\ v_2 = v_1 - v_3 = -v_3 \end{cases}$$

Therefore $V_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_2 = 1$.

- Following the same methodology, an easy calculation shows that $V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_3 = 2$.

Since the matrix A_1 admits 3 distinct eigenvalues with multiplicity 1, it is diagonalizable on $\mathbb{R}$, and

$$A_1 = PDP^{-1} \quad \text{with} \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad P = \big(V_1|V_2|V_3\big) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

The general solution of the differential system $Y' = A_1 Y$ is then given by

$$Y(t) = K_1 e^{\lambda_1 t} V_1 + K_2 e^{\lambda_2 t} V_2 + K_3 e^{\lambda_3 t} V_3 = K_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + K_2 e^t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + K_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

with $K_1,\ K_2,\ K_3 \in \mathbb{R}$.

2. For the matrix A_2, we calculate

$$\chi_{A_2}(X) = -X^2(X-6)$$

Hence, A_2 admits two eigenvalues $\lambda_1 = 0$ (with algebraic multiplicity 2), and $\lambda_2 = 6$ (with algebraic multiplicity 1).
Let V be an eigenvector associated to the eigenvalue $\lambda_1 = 0$, it satisfies

$$\begin{cases} v_1 + 2v_2 - v_3 = 0 \\ 2v_1 + 4v_2 - 2v_3 = 0 \\ -v_1 - 2v_2 + v_3 = 0 \end{cases}$$

Since $L_3 = -L_1$ and $L_2 = 2L_1$, the system reduces to a single equation

$$v_1 + 2v_2 - v_3 = 0,$$

in other words, the eigenspace associated to λ_1 is a subspace of $\mathbb{R}^3$ of dimension 2. One can check that the vectors

$$V_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

form a basis of this subspace (there are linearly independent and solve the previous equation).
We easily determine an eigenvector V_3 associated to $\lambda_2 = 6$: it solves the linear system

$$\begin{cases} v_1 + 2v_2 - v_3 = 6v_1 \\ 2v_1 + 4v_2 - 2v_3 = 6v_2 \end{cases}$$

Calculating $L_2 - 2L_1$ we get

$$6(v_2 - 2v_1) = 0 \qquad \Longrightarrow \qquad v_2 = 2v_1.$$

Replacing in L_1 we get $5v_1 - v_3 = 6v_1$, i.e. $v_3 = -v_1$. An eigenvector associated to $\lambda_2 = 6$ is thus $V_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.
Since the dimension of the eigenspace associated to the eigenvalue $\lambda_1 = 0$ is 2 which is the algebraic multiplicity of λ_1, the family (V_1, V_2, V_3) is a basis of $\mathbb{R}^3$ and A_2 is diagonalizable:

$$A_2 = PDP^{-1} \quad \text{with} \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad P = \big(V_1|V_2|V_3\big) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

The general solution of the differential system $Y' = A_2Y$ is then given by

$$Y(t) = K_1e^{\lambda_1 t}V_1 + K_2e^{\lambda_1 t}V_2 + K_3e^{\lambda_2 t}V_3 = K_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + K_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + K_3e^{6t} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

with $K_1,\ K_2,\ K_3 \in \mathbb{R}$.