Aix-Marseille Université — Master CMB-Bio
2019-2020

Continuous dynamical systems and modeling
Linear ODEs - Stability

Exercise 1. For the following matrices

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

- find the general solution of the differential system $Y' = AY$;
- determine the long-time behavior of the solution and the nature of the equilibrium point $Y^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;
- sketch the phase portrait.

Correction.

(1) We consider the matrix

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$$

for which

$$\mathrm{Tr}(A) = -3, \quad \det(A) = 2.$$

The characteristic polynomial of A is then

$$P_A(X) = X^2 - \mathrm{Tr}(A)X + \det(A) = X^2 + 3X + 2.$$

Since the discriminant of this polynomial is

$$\Delta = \mathrm{Tr}(A)^2 - 4\det(A) = 1 > 0,$$

the polynomial admits two real roots, $\lambda_{1,2}$, given by

$$\lambda_1 = \frac{\mathrm{Tr}(A) - \sqrt{\Delta}}{2} = -2, \quad \lambda_2 = \frac{\mathrm{Tr}(A) + \sqrt{\Delta}}{2} = -1.$$

λ_1, λ_2 are the eigenvalues of the matrix A. Since their multiplicity is one, the matrix A is diagonalizable. Let us determine eigenvectors, V_1, V_2, associated to these eigenvalues: An eigenvector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ associated to λ_1 satisfies

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

i.e.

$$\begin{cases} -3x_1 + 2x_2 = -2x_1 \\ -x_1 = -2x_2 \end{cases}$$

system which reduces to the single equation (the algebraic multiplicity of λ_1 is 1, so its geometric multiplicity, that is the dimension of the associated eigenspace, is necessarily 1):

$$x_1 = 2x_2.$$

An eigenvector associated to λ_1 is then $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and the eigenspace associated to λ_1: $E_1 = \text{Span}\,\{V_1\}$.
In the same manner, an eigenvector associated to λ_2 is $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the eigenspace associated to λ_2: $E_2 = \text{Span}\,\{V_2\}$.

Since A is diagonalizable, the general form of the solution to the differential system $Y' = AY$ is then

$$Y(t) = K_1 e^{\lambda_1 t} V_1 + K_2 e^{\lambda_2 t} V_2 = K_1 e^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + K_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for some constants $K_1, K_2 \in \mathbb{R}$ which are determined by the initial condition.
As $t \to +\infty$, since both eigenvalues are negative, $Y(t) \to 0$ and all the trajectories converge to the origin $Y^* = (0, 0)$. The equilibrium point is *asymptotically stable*, we also say that it is a *sink point.*
More precisely, since $\lambda_2 > \lambda_1$, the asymptotic behavior of $Y(t)$ is governed by the eigenvector V_2:

$$Y(t) \underset{t \to +\infty}{\sim} K_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A representation of the phase portrait is given in Figure 1.

(2) We consider the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

for which

$$\text{Tr}(A) = -1, \quad \det(A) = -1.$$

The characteristic polynomial of A is then

$$P_A(X) = X^2 - \text{Tr}(A)X + \det(A) = X^2 + X - 1.$$

Since the discriminant of this polynomial is

$$\Delta = \text{Tr}(A)^2 - 4\det(A) = 5 > 0,$$

the polynomial admits two real roots, $\lambda_{1,2}$, given by

$$\lambda_1 = \frac{\text{Tr}(A) - \sqrt{\Delta}}{2} = \frac{-1 - \sqrt{5}}{2} < 0, \quad \lambda_2 = \frac{\text{Tr}(A) + \sqrt{\Delta}}{2} = \frac{-1 + \sqrt{5}}{2} > 0.$$

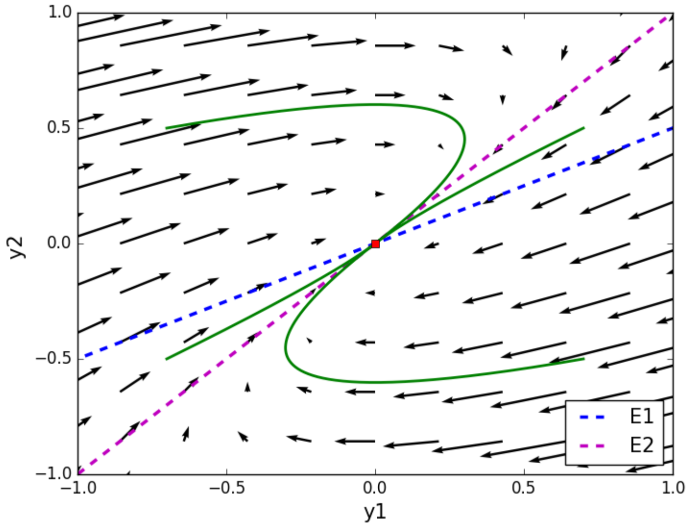

FIGURE 1. Phase portrait, case 1: sink point

$\lambda_1,\ \lambda_2$ are the eigenvalues of the matrix A. Since their multiplicity is one, the matrix A is diagonalizable. Let us determine eigenvectors, V_1, V_2, associated to these eigenvalues: An eigenvector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ associated to λ_1 satisfies

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

i.e.

$$\begin{cases} -x_1 + x_2 = \lambda_1 x_1 \\ x_1 = \lambda_1 x_2 \end{cases}$$

system which reduces to the single equation (the algebraic multiplicity of λ_1 is 1, so its geometric multiplicity, that is the dimension of the associated eigenspace, is necessarily 1):

$$x_1 = \lambda_1 x_2.$$

An eigenvector associated to λ_1 is then $V_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ and the eigenspace associated to λ_1: $E_1 = \text{Span}\,\{V_1\}$.

In the same manner, an eigenvector associated to λ_2 is $V_1 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$ and the eigenspace associated to λ_2: $E_2 = \text{Span}\,\{V_2\}$.

Since A is diagonalizable, the general form of the solution to the differential system $Y' = AY$ is

$$Y(t) = K_1 e^{\lambda_1 t} V_1 + K_2 e^{\lambda_2 t} V_2 = K_1 e^{\lambda_1 t} \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} + K_2 e^{\lambda_2 t} \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

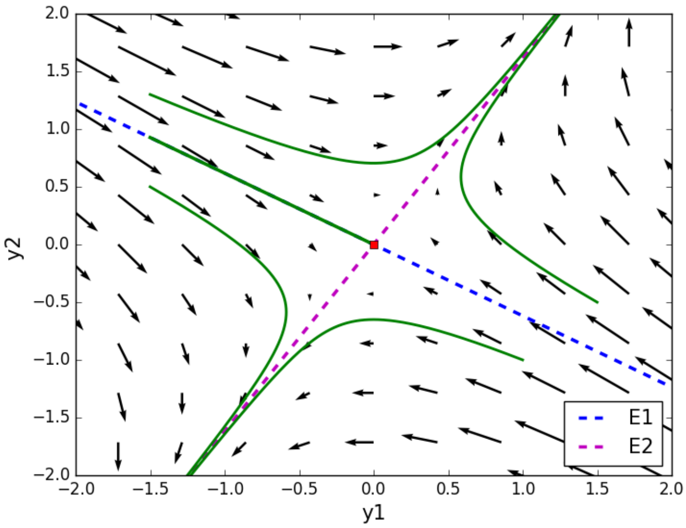

FIGURE 2. Phase portrait, case 2: saddle point

for some constants $K_1, K_2 \in \mathbb{R}$ which are determined by the initial condition. Namely, taking $t = 0$ in the previous equation yields

$$Y_0 = K_1 \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} + K_2 \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}.$$

As $t \to +\infty$, since λ_2 is positive and λ_1 negative, the equilibrium point $Y^* = (0,0)$ is a *saddle point*. The stable subspace is $E^s = E_1 = \mathrm{Span}\,\{V_1\}$, and the unstable space $E^u = E_2 = \mathrm{Span}\,\{V_2\}$. More precisely, it means that, except for initial data on E_1 (for which Y converge to Y^*), for all other initial data the associated solution Y goes to $+\infty$ as $t \to +\infty$: we have

$$Y(t) \underset{t\to+\infty}{\sim} K_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{if} \quad Y_0 = K_1 V_1 + K_2 V_2 \quad \text{with} \quad K_2 \neq 0.$$

A representation of the phase portrait is given in Figure 2.

(3) We consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for which

$$\mathrm{Tr}(A) = 0, \quad \det(A) = 1.$$

The characteristic polynomial of A is then

$$P_A(X) = X^2 - \mathrm{Tr}(A)X + \det(A) = X^2 + 1.$$

Since the associated discriminant is

$$\Delta = \mathrm{Tr}(A)^2 - 4\det(A) = -4 < 0,$$

the polynomial admits two complex conjugate roots, $\lambda_{1,2}$, given by

$$\lambda_1 = \frac{\mathrm{Tr}(A) + i\sqrt{-\Delta}}{2} = i, \quad \lambda_2 = \frac{\mathrm{Tr}(A) - i\sqrt{-\Delta}}{2} = -i = \overline{\lambda_1}.$$

λ_1, λ_2 are the eigenvalues of the matrix A. Since their multiplicity is one, the matrix A is diagonalizable (on $\mathbb{C}$). Let us determine an eigenvector, V_1, associated to $\lambda_1 = i$. It satisfies

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

system which reduces to the single equation:

$$x_1 = ix_2.$$

An eigenvector associated to λ_1 is then $V_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and the eigenspace associated to λ_1: $E_1 = \mathrm{Span}\{V_1\}$.

Since A is diagonalizable on $\mathbb{C}$, a real solution Y to the differential system $Y' = AY$ takes the general form

$$\begin{aligned} Y(t) &= K_1 \mathcal{R}e\left(e^{\lambda_1 t} V_1\right) + K_2\, \mathcal{I}m\left(e^{\lambda_1 t} V_1\right) \\ &= K_1 \mathcal{R}e\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}\right) + K_2\, \mathcal{I}m\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}\right), \qquad \text{for some } K_1, K_2 \in \mathbb{R}. \end{aligned}$$

We recall that for a complex number $z \in \mathbb{C}$, $\mathcal{R}e(z)$ and $\mathcal{I}m(z)$ denote respectively the real and the imaginary part of z, namely

$$z = \mathcal{R}e(z) + i\,\mathcal{I}m(z).$$

Let us also recall that for $z = e^{i\theta}$, we have

$$\mathcal{R}e(z) = \cos(\theta), \quad \mathcal{I}m(z) = \sin(\theta) \quad \text{that is} \quad e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Therefore

$$\begin{aligned} e^{\lambda_1 t} V_1 &= e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} \\ &= \big(\cos(t) + i\sin(t)\big)\begin{pmatrix} i \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} i\big(\cos(t) + i\sin(t)\big) \\ \cos(t) + i\sin(t) \end{pmatrix} \\ &\overset{(i^2=-1)}{=} \begin{pmatrix} -\sin(t) + i\cos(t) \\ \cos(t) + i\sin(t) \end{pmatrix} \\ &= \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + i\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \end{aligned}$$

and thus

$$\mathcal{R}e\left(e^{\lambda_1 t} V_1\right) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}, \qquad \mathcal{I}m\left(e^{\lambda_1 t} V_1\right) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.$$

As a consequence, the general solution writes

$$Y(t) = K_1 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + K_2 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \qquad K_1, K_2 \in \mathbb{R}.$$

Since the functions $t \mapsto \cos(t)$ and $t \mapsto \sin(t)$ are bounded functions, all trajectories remain in a neighborhood of the origin, we say that the equilibrium point $Y^* = (0,0)$ is *stable*, or that it is a *center point*.
More precisely, let us show that the trajectories are here circles centered at $(0,0)$. At time $t = 0$, we have

$$Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = K_1 \begin{pmatrix} -\sin(0) \\ \cos(0) \end{pmatrix} + K_2 \begin{pmatrix} \cos(0) \\ \sin(0) \end{pmatrix} = K_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + K_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and then

$$K_1 = y_2(0), \quad K_2 = y_1(0).$$

Passing to the polar coordinates (r, θ) (roughly speaking r denotes the distance between Y and the origin $(0,0)$, and θ the angle with the horizontal axis) which are such that

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} r(t)\cos(\theta(t)) \\ r(t)\sin(\theta(t)) \end{pmatrix}$$

we have

$$\begin{aligned}
(r(t))^2 &= (y_1(t))^2 + (y_2(t))^2 \\
&= \big(-\sin(t)K_1 + \cos(t)K_2\big)^2 + \big(\cos(t)K_1 + \sin(t)K_2\big)^2 \\
&= K_1^2 \sin^2(t) + K_2^2 \cos^2(t) - 2K_1K_2 \sin(t)\cos(t) \\
&\quad + K_1^2 \cos^2(t) + K_2^2 \sin^2(t) + 2K_1K_2 \sin(t)\cos(t) \\
&= K_1^2(\sin^2(t) + \cos^2(t)) + K_2^2(\sin^2(t) + \cos^2(t)) \\
&= K_1^2 + K_2^2 \qquad\qquad \text{since } \sin^2(t) + \cos^2(t) = 1 \\
&= (y_2(0))^2 + (y_1(0))^2 \\
&= (r(0))^2.
\end{aligned}$$

Hence, the distance to $(0,0)$ is constant in time, which means that the trajectories are circles.
To determine the sense of the rotation, let us first observe that

$$y_1(t) = r(t)\cos(\theta(t)) = r(0)\cos(\theta(t))$$

since we have proved that r was constant in time. Differentiating with respect to time and using the chain rule, we get

$$y_1'(t) = -r(0)\sin(\theta(t))\theta'(t).$$

On the other hand, the differential system $Y'(t) = AY(t)$ (we consider the first line) gives that

$$y_1'(t) = -y_2(t) = -r(t)\sin(\theta(t)) = -r(0)\sin(\theta(t)).$$

Therefore, we have the equality:

$$-r(0)\sin(\theta(t))\theta'(t) = -r(0)\sin(\theta(t))$$

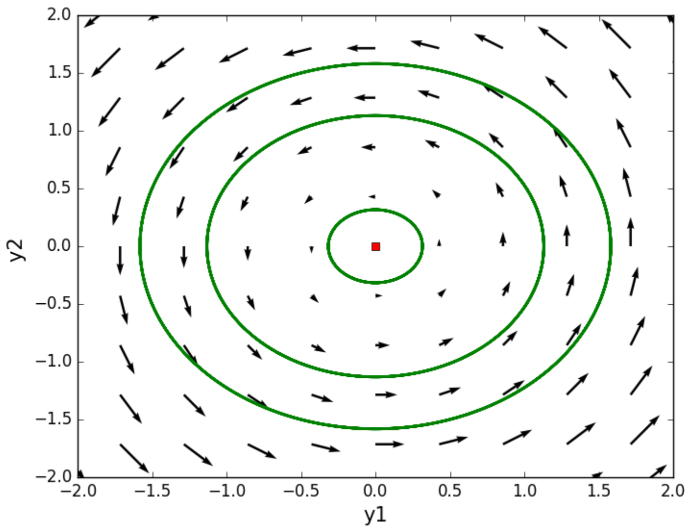

FIGURE 3. Phase portrait, case 3: center point

which implies that

$$\theta'(t) = 1.$$

Namely the angle θ increases with time and the sign of the rotation is positive (trigonometric, or counterclockwise, direction). A representation of the phase portrait is given in Figure 3.

(4) We consider the matrix

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

for which

$$\mathrm{Tr}(A) = -1, \quad \det(A) = 1.$$

The characteristic polynomial of A is then

$$P_A(X) = X^2 - \mathrm{Tr}(A)X + \det(A) = X^2 + X + 1.$$

Since the associated discriminant is

$$\Delta = \mathrm{Tr}(A)^2 - 4\det(A) = -3 < 0,$$

the polynomial admits two complex conjugate roots, $\lambda_{1,2}$, given by

$$\lambda_1 = \frac{\mathrm{Tr}(A) + i\sqrt{-\Delta}}{2} = \frac{-1 + i\sqrt{3}}{2}, \quad \lambda_2 = \frac{\mathrm{Tr}(A) - i\sqrt{-\Delta}}{2} = \frac{-1 - i\sqrt{3}}{2} = \overline{\lambda_1}.$$

λ_1, λ_2 are the eigenvalues of the matrix A. Since their multiplicity is one, the matrix A is diagonalizable (on $\mathbb{C}$). One can check that

$$V_1 = \begin{pmatrix} -\lambda_1 \\ 1 \end{pmatrix}$$

is an eigenvector associated to λ_1.

Since A is diagonalizable on $\mathbb{C}$, a real solution Y to the differential system $Y' = AY$ takes the general form

$$\begin{aligned} Y(t) &= K_1 \mathcal{R}e\left(e^{\lambda_1 t} V_1\right) + K_2 \mathcal{I}m\left(e^{\lambda_1 t} V_1\right) \\ &= K_1 \mathcal{R}e\left(e^{\frac{-1+i\sqrt{3}}{2}t} \begin{pmatrix} \frac{1-i\sqrt{3}}{2} \\ 1 \end{pmatrix}\right) + K_2 \mathcal{I}m\left(e^{\frac{-1+i\sqrt{3}}{2}t} \begin{pmatrix} \frac{1-i\sqrt{3}}{2} \\ 1 \end{pmatrix}\right) \end{aligned}$$

for some $K_1, K_2 \in \mathbb{R}$. We calculate

$$\begin{aligned} & e^{\lambda_1 t} V_1 \\ = \ & e^{\frac{-1+i\sqrt{3}}{2}t} \begin{pmatrix} \frac{1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} \\ = \ & e^{-\frac{t}{2}} e^{\frac{i\sqrt{3}}{2}t} \begin{pmatrix} \frac{1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} \\ = \ & e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}t\right) + i \sin\left(\frac{\sqrt{3}}{2}t\right)\right) \begin{pmatrix} \frac{1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} \\ = \ & e^{-\frac{t}{2}} \begin{pmatrix} \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + i \sin\left(\frac{\sqrt{3}}{2}t\right)\right) \\ \cos\left(\frac{\sqrt{3}}{2}t\right) + i \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} \\ = \ & e^{-\frac{t}{2}} \begin{pmatrix} \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) + i\left(\frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right) \\ \cos\left(\frac{\sqrt{3}}{2}t\right) + i \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} \\ = \ & e^{-\frac{t}{2}} \begin{pmatrix} \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ \cos\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} + ie^{-\frac{t}{2}} \begin{pmatrix} \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) \\ \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} \end{aligned}$$

and thus

$$V_r := \mathcal{R}e\left(e^{\lambda_1 t} V_1\right) = e^{-\frac{t}{2}} \begin{pmatrix} \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ \cos\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix},$$

$$V_i := \mathcal{I}m\left(e^{\lambda_1 t} V_1\right) = e^{-\frac{t}{2}} \begin{pmatrix} \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) \\ \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}.$$

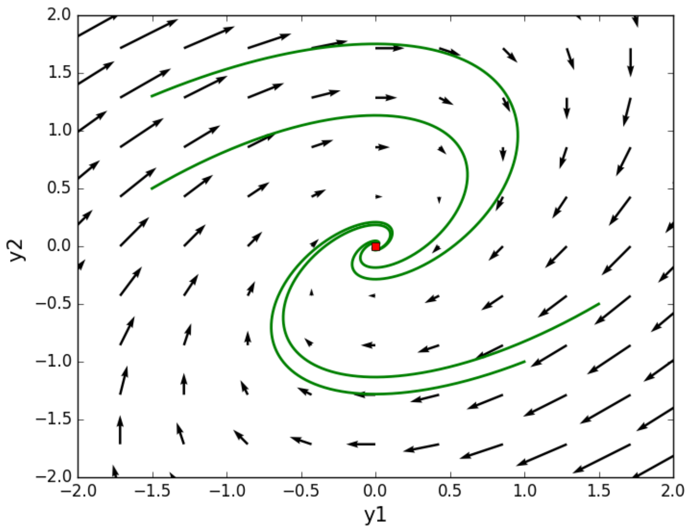

FIGURE 4. Phase portrait, case 4: spiral sink point

As a consequence, the general solution writes

$$\begin{aligned} Y(t) &= K_1 V_r + K_2 V_i \\ &= e^{-\frac{t}{2}} \Bigg[K_1 \begin{pmatrix} \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ \cos\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} \\ &\qquad + K_2 \begin{pmatrix} \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) \\ \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} \Bigg], \qquad K_1, K_2 \in \mathbb{R}. \end{aligned}$$

Due to the exponential factor, all the solutions Y converge to $(0,0)$ as $t \to 0$. The cosinus and sinus functions make the solution turn around the origin. The equilibrium point $Y^* = (0,0)$ is *asymptotically stable*, it is a *spiral sink point*. A representation of the trajectories is given in Figure 4.

Additional comment. The determination of the rotational direction by passage to polar coordinates is more complicated than in the previous case. But, thanks to the simple form of the matrix, it is not too difficult to guess it: consider the initial point $Y(t_0) = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have

$$\begin{pmatrix} y_1'(t_0) \\ y_2'(t_0) \end{pmatrix} = A \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \end{pmatrix}$$

and therefore

$$y_1'(t_0) = -y_1(t_0) = -1, \quad y_2'(t_0) = -y_1(t_0) = -1.$$

This implies that the trajectories turn clock-wisely around the origin as we observe in Figure 4.