An introduction to Shallow Water Equations

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Practical informations

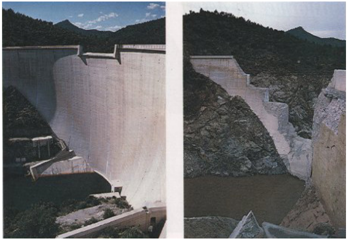
- Lectures with me, TP with Thierry Gallouët
- Evaluation
  - 1/3* CC + 2/3*Exam
  - CC = TP + Project

Mathematics and fluid mechanics - Motivations

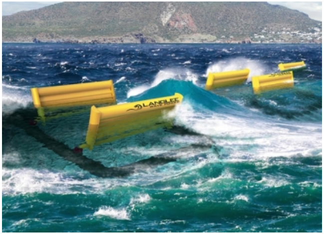
- nearshore hydrodynamics
  - coastal morpho-dynamics and erosion
  - flood-risk prevention

Mathematics and fluid mechanics - Motivations

- hydroelectric dam



- wave energy converters

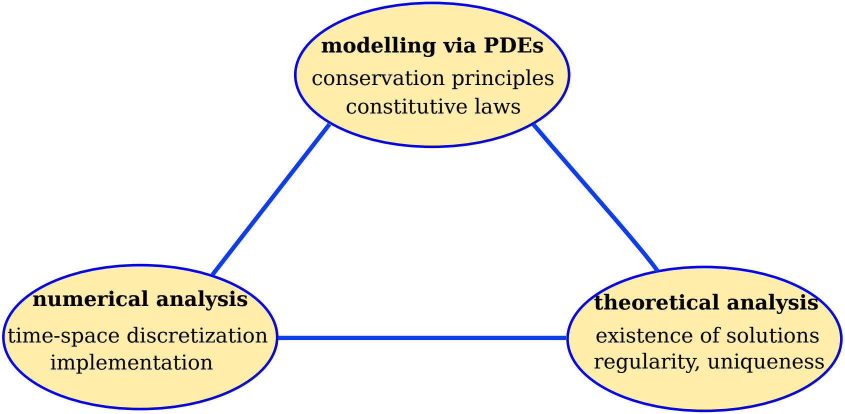


Mathematics and fluid mechanics - More complex flows

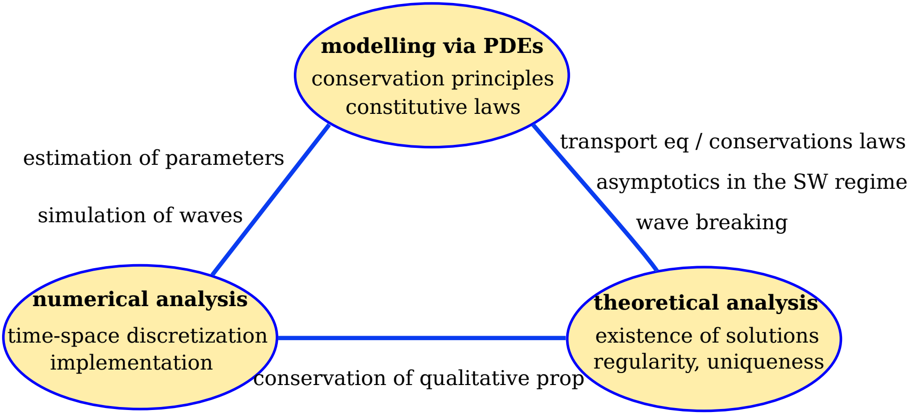
- avalanches

- pyroclastic flows

Mathematics and Fluid Mechanics



Shallow Water (SW) equations

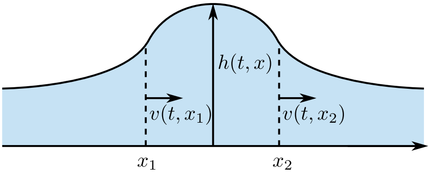


Outline

1. Classical fluid equations and the Shallow Water regime
2. Theoretical analysis of the Saint-Venant system
    1. Classical (regular) solutions
    2. Weak solutions
3. Numerical approximation of the solutions of the Saint-Venant system

Modelling fluid systems and the Shallow Water regime

Modelling water waves in shallow water - first approach



- conservation of the volume of fluid between x_1 and x_2

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} h(t,x)\,\mathrm{d}x = h(t,x_1)v(t,x_1) - h(t,x_2)v(t,x_2)$$

$$\Longrightarrow \quad \int_{x_1}^{x_2} \partial_t h(t,x)\,\mathrm{d}x = -\int_{x_1}^{x_2} \partial_x(hv)\,\mathrm{d}x \qquad \forall\ x_1,\ x_2$$

- **conservation law**

$$\partial_t h + \partial_x(hv) = 0$$

- we close the system with an ad hoc **velocity law** $v = v(h) = h$

$$\Longrightarrow \quad \boxed{\partial_t h + \partial_x(h^2) = 0} \quad \text{Burgers equation}$$

Macroscopic modelling - variables describing the flow

- **velocity**:
  - Eulerian standpoint: let t be a given observation time, $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ a given position $u(t, \mathbf{x}) \in \mathbb{R}^{\mathbf{d}}$ is the average velocity of the fluid particle located at $\mathbf{x}$ at time t
  - Lagrangian standpoint: given a fluid particle located at position $\mathbf{y}$ at $t = 0$, we define its trajectory (or characteristics) $t \mapsto X_{\mathbf{y}}(t)$ and the Lagrangian velocity $U(t, \mathbf{y})$

$$U(t, \mathbf{y}) = \frac{\mathrm{d}\mathbf{X}_{\mathbf{y}}(\mathbf{t})}{\mathrm{d}\mathbf{t}} = \mathbf{u}\big(\mathbf{t}, \mathbf{X}_{\mathbf{y}}(\mathbf{t})\big)$$

- **density**:

  $\rho(t, \mathbf{x}) \in \mathbb{R}$ represents the nb of micro. elements in the fluid particle located at $(t, \mathbf{x})$

  we shall assume that the density of the fluid is constant

Other possible variables: temperature, salinity, etc.

Conservation principles - Constitutive laws

- **Conservation of mass** for any $\omega \subset \Omega$, we denote n the outer normal to $\partial\omega$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\omega \rho \,\mathrm{d}x\right) = -\int_{\partial\omega} \rho\,(u\cdot n)\,\mathrm{d}\sigma \underset{\substack{\uparrow \\ \text{Green-Ostrogradsky}}}{=} -\int_\omega \mathrm{div}\,(\rho u)\,\mathrm{d}x$$

recall: let $u = (u^1, \dots, u^n)$, $\mathrm{div}\, u = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} u^i$

$\rightarrow$ local conservation of mass $\boxed{\partial_t \rho + \mathrm{div}\,(\rho u) = 0}$

if $\rho \equiv \mathrm{cst} \rightsquigarrow$ incompressibility condition $\boxed{\mathrm{div}\, u = 0}$

- **Balance of momentum**: expression of Newton's second law

$$\boxed{\rho\Big(\partial_t u + (u\cdot\nabla)u\Big) = \operatorname{div}\mathbb{T} + \rho f}$$

$\mathbb{T}$: Cauchy stress, internal forces acting on the fluid
f: external forces

other formulation (using the mass conservation):

$$\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) = \operatorname{div}\mathbb{T} + \rho f$$

- **Constitutive laws**:

$$\mathbb{T} = 2\mu \mathrm{D}(u) + (\lambda \operatorname{div} u - p)\,\mathrm{Id}$$

where $\mathrm{D}(u) = \dfrac{\nabla u + (\nabla u)^T}{2} \in \mathbb{R}^d\times\mathbb{R}^d$, $\quad \mu,\lambda$ are the viscosities of the fluid

$p = p(\rho)$ is the pressure of the fluid (ex: $p(\rho) = a\rho^\gamma$)

Incompressible Navier-Stokes (/ Euler) equations

$$\begin{cases} \operatorname{div} u = 0 \\ \partial_t u + (u \cdot \nabla) u + \nabla \bar{p} - 2\nu \operatorname{div}(\mathrm{D}(u)) = f \end{cases}$$

incompressible Euler system if the viscosity $\nu = \frac{\mu}{\rho} = 0$

$\bar{p}$: Lagrange multiplier associated to the incompressibility constraint $\operatorname{div} u = 0$

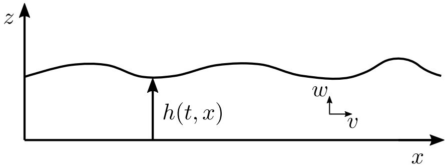
in the following, we shall assume that <u>only gravity acts on the fluid</u>

2D Free surface flows, $u = (v, w)$

$$\partial_x v + \partial_z w = 0 \qquad \leftarrow \text{ incomp. constraint}$$

$$\partial_t v + \partial_x v^2 + \partial_z (wv) + \partial_x \bar{p} - 2\nu\partial_{xx} v - \nu\partial_{zz} v - \nu\partial_{xz} w = 0 \qquad \leftarrow x\text{-momentum eq}$$

$$\partial_t w + \partial_x (vw) + \partial_z w^2 + \partial_z \bar{p} - \nu\partial_{xz} v - \nu\partial_{xx} w - 2\nu\partial_{zz} w = -g \qquad \leftarrow z\text{-momentum eq}$$



Boundary conditions

- at the bottom $z = 0$:
  - no-penetration $w_{|z=0} = 0$
  - wall-law (Navier's condition with no friction) $\partial_z v_{|z=0} + \partial_x w_{|z=0} = 0$
- continuity of the stress tensor at $z = h$: $\big(2\nu \mathrm{D}(u) - p \ \mathrm{Id}\big).\mathbf{n} = -p_{\mathrm{atm}}\mathbf{n}$

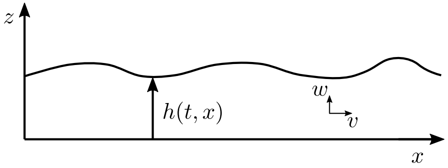
where $\mathbf{n}$ is the unit outward normal to the surface: $\mathbf{n} = \dfrac{1}{\sqrt{1 + (\partial_x h)^2}} \begin{pmatrix} -\partial_x h \\ 1 \end{pmatrix}$

2D Free surface flows, $u = (v, w)$

$$\partial_x v + \partial_z w = 0 \qquad \leftarrow \text{ incomp. constraint}$$

$$\partial_t v + \partial_x v^2 + \partial_z(wv) + \partial_x \bar{p} - 2\nu\partial_{xx} v - \nu\partial_{zz} v - \nu\partial_{xz} w = 0 \qquad \leftarrow \ x\text{-momentum eq}$$

$$\partial_t w + \partial_x(vw) + \partial_z w^2 + \partial_z \bar{p} - \nu\partial_{xz} v - \nu\partial_{xx} w - 2\nu\partial_{zz} w = -g \qquad \leftarrow \ z\text{-momentum eq}$$



Boundary conditions

- at the bottom $z = 0$:
  - no-penetration $w_{|z=0} = 0$
  - wall-law (Navier's condition with no friction) $\partial_z v_{|z=0} + \partial_x w_{|z=0} = 0$
- at the free surface $z = h(t, x)$:

$$\begin{cases} (\bar{p}_{|z=h} - p_{\mathrm{atm}})\partial_x h + \nu\big(\partial_z v_{|z=h} + \partial_x w_{|z=h} - 2\partial_x v_{|z=h}\partial_x h\big) = 0 \\ \bar{p}_{|z=h} - p_{\mathrm{atm}} + \nu\big(\partial_z v_{|z=h}\partial_x h + \partial_x w_{|z=h}\partial_x h - 2\partial_z w_{|z=h}\big) = 0 \end{cases}$$

Conservation of mass

We introduce the indicator function of the fluid region

$$\varphi(t,x,z) = \begin{cases} 1 & \text{if } z \in [0, h(t,x)] \\ 0 & \text{otherwise} \end{cases}$$

- the fluid is transported at velocity $u = (v, w)$

$$\partial_t \varphi + v\partial_x \varphi + w\partial_z \varphi = 0 \underset{\text{incomp. cond.}}{\Longrightarrow} \boxed{\partial_t \varphi + \partial_x(v\varphi) + \partial_z(w\varphi) = 0}$$

- integration over $z \in [0, +\infty[$

$$\begin{aligned} 0 &= \int_0^\infty \partial_t \varphi \,\mathrm{d}z + \int_0^\infty \Big[\partial_x(v\varphi) + \partial_z(w\varphi)\Big]\,\mathrm{d}z \\ &= \partial_t \left(\int_0^\infty \varphi \,\mathrm{d}z\right) + \partial_x \left(\int_0^\infty (v\varphi)\,\mathrm{d}z\right) - \underbrace{[w\varphi]_{|z=0}}_{=0} \\ &= \partial_t h + \partial_x \left(\int_0^{h(t,x)} v \,\mathrm{d}z\right) \end{aligned}$$

$$\boxed{\partial_t h + \partial_x(h\bar{v}) = 0} \quad \text{with} \quad \bar{v}(t,x) = \frac{1}{h(t,x)} \left(\int_0^{h(t,x)} v(t,z)\,\mathrm{d}z\right)$$

Kinematic condition at the free surface

Coming back to the transport equation satisfied by φ

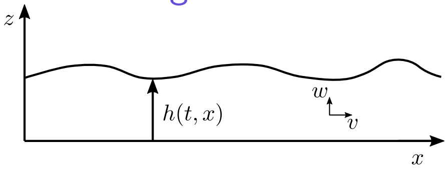
$$\partial_t \varphi + \partial_x(v\varphi) + \partial_z(w\varphi) = 0$$

- integration over the height of the fluid $z \in [0, h(t,x)]$

$$\begin{aligned}
0 &= \int_0^{h(t,x)} \partial_t \varphi \, \mathrm{d}z + \int_0^{h(t,x)} \Big[\partial_x(v\varphi) + \partial_z(w\varphi)\Big] \, \mathrm{d}z \\
&= \partial_t \left(\int_0^{h(t,x)} \varphi \, \mathrm{d}z \right) - \partial_t h \; \varphi_{|z=h} + \partial_x \left(\int_0^{h(t,x)} v \, \mathrm{d}z \right) \\
&\qquad - [v\varphi]_{|z=h} \partial_x h + [w\varphi]_{|z=h} - \underbrace{[w\varphi]_{|z=0}}_{=0} \\
&= \underbrace{\partial_t h + \partial_x(h\bar{v})}_{=0} - \partial_t h - v_{|z=h} \partial_x h + w_{|z=h}
\end{aligned}$$

$$\boxed{\partial_t h + v_{|z=h} \partial_x h - w_{|z=h} = 0} \quad \text{kinematic equation}$$

Scaling - Shallow water regime



- H: characteristic height of the flow, L: characteristic length in the x-direction

$$\boxed{\varepsilon := \frac{H}{L} \ll 1}$$

- other characteristic dimensions (recall that the density is equal to 1)

$$V, \quad W = \varepsilon V, \quad T = \frac{L}{V}, \quad P = V^2$$

- dimensionless quantities

$$\widetilde{v} = \frac{v}{V}, \quad \widetilde{w} = \frac{w}{W}, \quad \widetilde{x} = \frac{x}{L}, \quad \widetilde{z} = \frac{z}{H}, \quad \widetilde{t} = \frac{t}{T}, \quad \widetilde{p} = \frac{p}{P}$$

characteristic numbers: Reynolds nb $\mathrm{Re} := \dfrac{VL}{\nu}$, Froude nb $\mathrm{Fr} := \dfrac{V}{\sqrt{gH}}$

Shallow water regime - rescaled Navier-Stokes equations

$$\partial_{\tilde{x}}\tilde{v} + \partial_{\tilde{z}}\tilde{w} = 0$$

$$\partial_{\tilde{t}}\tilde{v} + \partial_{\tilde{x}}\tilde{v}^2 + \partial_{\tilde{z}}(\tilde{w}\tilde{v}) + \partial_{\tilde{x}}\tilde{p} = \frac{1}{\mathrm{Re}}\left(2\partial_{xx}v + \frac{1}{\varepsilon^2}\partial_{\tilde{z}\tilde{z}}\tilde{v} + \partial_{\tilde{x}\tilde{z}}\tilde{w}\right)$$

$$\varepsilon\left(\partial_{\tilde{t}}\tilde{w} + \partial_{\tilde{x}}(\tilde{v}\tilde{w}) + \partial_{\tilde{z}}\tilde{w}^2\right) + \frac{1}{\varepsilon}\partial_{\tilde{z}}\tilde{p} = -\frac{1}{\varepsilon \mathrm{Fr}^2} + \frac{1}{\varepsilon \mathrm{Re}}\left(\partial_{\tilde{x}\tilde{z}}\tilde{v} + \varepsilon^2\partial_{\tilde{x}\tilde{x}}\tilde{w} + 2\partial_{\tilde{z}\tilde{z}}\tilde{w}\right)$$

Boundary conditions

- at the bottom $\tilde{z} = 0$:
  - no-penetration $\tilde{w}_{|\tilde{z}=0} = 0$
  - wall-law (Navier's condition with no friction) $\frac{1}{\varepsilon}\partial_{\tilde{z}}\tilde{v}_{|\tilde{z}=0} + \varepsilon\partial_{\tilde{x}}\tilde{w}_{|\tilde{z}=0} = 0$
- at the free surface $z = h(t, x)$:

$$\begin{cases} \varepsilon(\tilde{p}_{|z=h} - p_{\mathrm{atm}})\partial_{\tilde{x}}\tilde{h} + \frac{1}{\mathrm{Re}}\left(\frac{1}{\varepsilon}\partial_{\tilde{z}}\tilde{v}_{|\tilde{z}=\tilde{h}} + \varepsilon\partial_{\tilde{x}}\tilde{w}_{|\tilde{z}=\tilde{h}} - 2\varepsilon\partial_{\tilde{x}}\tilde{v}_{|\tilde{z}=\tilde{h}}\partial_{\tilde{x}}\tilde{h}\right) = 0 \\ \tilde{p}_{|z=h} - p_{\mathrm{atm}} + \frac{1}{\mathrm{Re}}\left(\partial_{\tilde{z}}\tilde{v}_{|\tilde{z}=\tilde{h}}\partial_{\tilde{x}}\tilde{h} + \varepsilon^2\partial_{\tilde{x}}\tilde{w}_{|\tilde{z}=\tilde{h}}\partial_{\tilde{x}}\tilde{h} - 2\partial_{\tilde{z}}\tilde{w}_{|\tilde{z}=\tilde{h}}\right) = 0 \end{cases}$$

Shallow water regime - rescaled Navier-Stokes equations

we drop all the $\tilde{\cdot}$

$$\partial_x v + \partial_z w = 0$$

$$\partial_t v + \partial_x v^2 + \partial_z(wv) + \partial_x p = \frac{1}{\mathrm{Re}}\left(2\partial_{xx} v + \frac{1}{\varepsilon^2}\partial_{zz} v + \partial_{xz} w\right)$$

$$\varepsilon^2\left(\partial_t w + \partial_x(vw) + \partial_z w^2\right) + \partial_z p = -\frac{1}{\mathrm{Fr}^2} + \frac{1}{\mathrm{Re}}\left(\partial_{xz} v + \varepsilon^2 \partial_{xx} w + 2\partial_{zz} w\right)$$

Boundary conditions

- at the bottom $z = 0$:
  - no-penetration $w_{|z=0} = 0$
  - wall-law $\partial_z v_{|z=0} + \varepsilon^2 \partial_x w_{|z=0} = 0$
- at the free surface $z = h(t,x)$:

$$\begin{cases} (p_{|z=h} - p_{\mathrm{atm}})\partial_x h + \frac{1}{\mathrm{Re}}\left(\frac{1}{\varepsilon^2}\partial_z v_{|z=h} + \partial_x w_{|z=h} - 2\partial_x v_{|z=h}\partial_x h\right) = 0 \\ p_{|z=h} - p_{\mathrm{atm}} + \frac{1}{\mathrm{Re}}\left(\partial_z v_{|z=h}\partial_x h + \varepsilon^2 \partial_x w_{|z=h}\partial_x h - 2\partial_z w_{|z=h}\right) = 0 \end{cases}$$

Shallow water regime - hydrostatic pressure

- z-momentum equation and conditions at the free surface

$$\varepsilon^2\left(\partial_t w + \partial_x(vw) + \partial_z w^2\right) + \partial_z p = -\frac{1}{\mathrm{Fr}^2} + \frac{1}{\mathrm{Re}}\left(\partial_{xz} v + \varepsilon^2 \partial_{xx} w + 2\partial_{zz} w\right)$$

$$\begin{cases} (p_{|z=h} - p_{\mathrm{atm}})\partial_x h + \frac{1}{\mathrm{Re}}\left(\frac{1}{\varepsilon^2}\partial_z v_{|z=h} + \partial_x w_{|z=h} - 2\partial_x v_{|z=h}\partial_x h\right) = 0 \\ p_{|z=h} - p_{\mathrm{atm}} + \frac{1}{\mathrm{Re}}\left(\partial_z v_{|z=h}\partial_x h + \varepsilon^2 \partial_x w_{|z=h}\partial_x h - 2\partial_z w_{|z=h}\right) = 0 \end{cases}$$

- neglecting the terms of order ε^2:

$$\begin{cases} \partial_z p = -\frac{1}{\mathrm{Fr}^2} + \frac{1}{\mathrm{Re}}\left(\partial_{xz} v + 2\partial_{zz} w\right) \\ \partial_z v_{|z=h} = 0 \\ p_{|z=h} - p_{\mathrm{atm}} - \frac{2}{\mathrm{Re}}\partial_z w_{|z=h} = 0 \end{cases}$$

Shallow water regime - hydrostatic pressure

$$\begin{cases} \partial_z p = -\dfrac{1}{\mathrm{Fr}^2} + \dfrac{1}{\mathrm{Re}} \left(\partial_{xz} v + 2\partial_{zz} w \right) \\ \partial_z v_{|z=h} = 0 \\ p_{|z=h} - p_{\mathrm{atm}} - \dfrac{2}{\mathrm{Re}} \partial_z w_{|z=h} = 0 \end{cases}$$

- integration for ξ between $h(t,x)$ and z

$$\begin{aligned} p(t,x,z) =& p(t,x,h) + \frac{1}{\mathrm{Fr}^2}(h-z) + \frac{1}{\mathrm{Re}} \int_h^z \partial_z \left(\partial_x v + 2\partial_z w \right) \mathrm{d}\xi \\ =& p_{\mathrm{atm}} + \frac{2}{\mathrm{Re}} \partial_z w_{|\xi=h} + \frac{1}{\mathrm{Fr}^2}(h-z) + \frac{1}{\mathrm{Re}} \left(\partial_x v_{|\xi=z} - \partial_x v_{|\xi=h} \right) \\ &+ \frac{2}{\mathrm{Re}} \left(\partial_z w_{|\xi=z} - \partial_z w_{|\xi=h} \right) \end{aligned}$$

- we finally use the incompressibility condition $\partial_z w = -\partial_x v$

$$\boxed{p(t,x,z) = p_{\mathrm{atm}} + \frac{1}{\mathrm{Fr}^2}(h-z) - \frac{1}{\mathrm{Re}} \left(\partial_x v_{|\xi=z} + \partial_x v_{|\xi=h} \right)}$$

if we neglect the viscosity $\rightarrow$ usual hydrostatic pressure $p(t,x,z) = p_{\mathrm{atm}} + \dfrac{1}{\mathrm{Fr}^2}(h-z)$

Vertical averaging of the x-momentum equation

$$\partial_t v + \partial_x v^2 + \partial_z(wv) + \partial_x p = \frac{1}{\mathrm{Re}}\left(2\partial_{xx} v + \frac{1}{\varepsilon^2}\partial_{zz} v + \partial_{xz} w\right) \qquad \leftarrow \text{ x-mom. eq}$$

$$\partial_t h + \partial_x h \; v_{|z=h} - w_{|z=h} = 0 \qquad \leftarrow \text{ kinematic cond.}$$

$$(p_{|z=h} - p_{\mathrm{atm}})\partial_x h + \frac{1}{\mathrm{Re}}\left(\frac{1}{\varepsilon^2}\partial_z v_{|z=h} + \partial_x w_{|z=h} - 2\partial_x v_{|z=h}\partial_x h\right) = 0$$

$$w_{|z=0} = 0, \quad \partial_z v_{|z=0} + \varepsilon^2 \partial_x w_{|z=0} = 0 \qquad \leftarrow \text{ cond. at } z = 0$$

- integration of the 1st eq between 0 and h using the boundary conditions

$$\partial_t\left(\int_0^h v\,\mathrm{d}z\right) + \partial_x\left(\int_0^h v^2\,\mathrm{d}z\right) + \left(\int_0^h \partial_x p\,\mathrm{d}z\right) - \frac{2}{\mathrm{Re}}\partial_x\left(\int_0^h \partial_x v\,\mathrm{d}z\right)$$

$$= \partial_t h v_{|z=h} + \partial_x h\,(v_{|z=h})^2 - w_{|z=h} v_{|z=h} + w_{|z=0} v_{|z=0}$$

$$+ \frac{1}{\mathrm{Re}}\left(-2\partial_x h \partial_x v_{|z=h} + \frac{1}{\varepsilon^2}\partial_z v_{|z=h} + \partial_x w_{|z=h}\right) + \frac{1}{\mathrm{Re}}\left(\frac{1}{\varepsilon^2}\partial_z v_{|z=0} + \partial_x w_{|z=0}\right)$$

$$= -(p_{|z=h} - p_{\mathrm{atm}})\partial_x h$$

Closure - The Saint-Venant system

$$\begin{cases} \partial_t h + \partial_x \left(\int_0^h v \, \mathrm{d}z \right) = 0 \\ \partial_t \left(\int_0^h v \, \mathrm{d}z \right) + \partial_x \left(\int_0^h v^2 \, \mathrm{d}z \right) + \left(\int_0^h \partial_x p \, \mathrm{d}z \right) \\ \qquad - \frac{2}{\mathrm{Re}} \partial_x \left(\int_0^h \partial_x v \, \mathrm{d}z \right) = -(p_{|z=h} - p_{\mathrm{atm}}) \partial_x h \end{cases}$$

- setting $\bar{v} = \frac{1}{h} \int_0^h v \, \mathrm{d}z$, the 1st equation rewrites

$$\partial_t h + \partial_x (h \bar{v}) = 0$$

- we assume that $\mathrm{Re} \sim O(\varepsilon^{-1})$ (viscosity very small)
- from the x-momentum eq. + boundary cond. at $z = h$:

$$\partial_{zz} v = O(\varepsilon), \quad \partial_z v_{|z=h} = O(\varepsilon)$$

$$\implies \partial_z v = O(\varepsilon) \implies v(t,x,z) = v(t,x,0) + O(\varepsilon)$$

$$\boxed{v(t,x,z) = \bar{v}(t,x) + O(\varepsilon)}$$ "motion by slices", uniform profile

$$\implies \overline{(v^2)} = \bar{v}^2 \quad \text{at first order}$$

Closure - The Saint-Venant system

$$\partial_t (h\bar{v}) + \partial_x \left(h\bar{v}^2\right) + \left(\int_0^h \partial_x p \,\mathrm{d}z\right) = -(p_{|z=h} - p_{\mathrm{atm}})\partial_x h + O(\varepsilon)$$

- at order ε: $p(t,x,z) = p_{\mathrm{atm}} + \dfrac{1}{\mathrm{Fr}^2}(h-z) + O(\varepsilon)$

$$\Longrightarrow \; p_{|z=h} = p_{\mathrm{atm}} + O(\varepsilon) \;\text{ and }\; \partial_x p = \frac{1}{\mathrm{Fr}^2}\partial_x h + O(\varepsilon)$$

- the resulting Saint-Venant system (at first order in ε) reads

$$\begin{cases} \partial_t h + \partial_x(h\bar{v}) = 0 \\ \partial_t(h\bar{v}) + \partial_x(h\bar{v}^2) + \dfrac{1}{2\mathrm{Fr}^2}\partial_x(h^2) = 0 \end{cases}$$

  - $h\bar{v}$: flow rate

- structural analogy with the compressible Euler equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p(\rho) = 0 \end{cases}$$

The Saint-Venant (Shallow Water) system - Summary

$$\Big[\text{2D incomp. Euler eq + BC (3 + 3 Eqs)}\Big] \longrightarrow \text{1D Shallow Water eq (2 Eqs)}$$

Recap:

- nondimensionalization of incompressible Euler equations
- hydrostatic approximation giving the pressure profile
- vertical averaging
- closure of the model by assuming a uniform velocity profile at leading order

$$\begin{cases} \partial_t h + \partial_x(h\bar{v}) = 0 \\ \partial_t(h\bar{v}) + \partial_x(h\bar{v}^2) + \dfrac{1}{2\mathrm{Fr}^2}\partial_x(h^2) = 0 \end{cases}$$

Extensions / Alternatives:

- other starting system (ex: Euler eq.), other forces (ex: Coriolis)
- other boundary conditions at the top (ex: $p_{\mathrm{atm}}(t,x)$, surface tension, etc.) or at the bottom (ex: topography, no-slip or friction law, etc.)
- other reference velocity profile (ex: linear, semi-parabolic)

Rmk: Large range of applications in geophysics

$\rightarrow$ models for avalanches, atmospheric flows, sediment transport, etc.

Some theoretical math. issues around Shallow Water eq.

- existence of solutions to the Shallow Water equations ?
  what regularity ? global existence / blow up in finite time ?

- rigorous justification of the asymptotics

Main references for the course

- D. Serre, *Systems of Conservation Laws* T.1
- L. C. Evans, *Partial Differential Equations*
- (C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*)

see also other master courses

- "Hyperbolic models for complex flows - application to sustainable energies" by E. Godlewski, J. Sainte-Marie
- "Shallow-water models for water waves" by V. Duchêne

and books, survey papers

- D. Lannes, *The Water Waves Problem, Mathematical Analysis and Asymptotics*
- D. Bresch, "Shallow Water equations and Related Topics"
- C. Mascia, "A dive into shallow water"

Propositions of projects

- format: work in pairs

  small report ($\sim$ 10-15 pages) + defense in March 15min (+ questions)

- possible subjects:
  - Complex flows in the shallow water regime
    $\rightarrow$ modeling of complex geophysical flows (avalanches, mud flows, etc.)
  - erosion model(s) $\rightarrow$ sediment transport, coastal morpho-dynamics
  - partially constrained free surface flows $\rightarrow$ subsurface flows, WEC models
  - viscous Saint-Venant equations
  - kinetic point of view on the Saint-Venant equations
  - ...