A CONVERGENT FV – FE SCHEME FOR THE STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract

In this paper, we prove a convergence result for a discretization of the three-dimensional stationary compressible Navier-Stokes equations assuming an ideal gas pressure law $p(\rho) = a\rho^\gamma$ with $\gamma > \frac{3}{2}$. It is the first convergence result for a numerical method with adiabatic exponents γ less than 3 when the space dimension is three. The considered numerical scheme combines finite volume techniques for the convection with the Crouzeix-Raviart finite element for the diffusion. A linearized version of the scheme is implemented in the industrial software CALIF^{3}S developed by the french *Institut de Radioprotection et de Sûreté Nucléaire* (IRSN).

Keywords: Stationary compressible Navier-Stokes equations, staggered discretization, finite volume - finite element method, convergence analysis.

Contents

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1 Introduction

Let Ω be an open bounded connected subset of $\mathbb{R}^d$, with $d = 2$ or 3, with Lipschitz boundary. We consider the system of stationary isentropic Navier-Stokes equations, posed for $\boldsymbol{x} \in \Omega$:

$$\operatorname{div}(\rho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$\mathbf{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \boldsymbol{\Delta} \boldsymbol{u} - (\mu + \lambda) \boldsymbol{\nabla}(\operatorname{div} \boldsymbol{u}) + a \boldsymbol{\nabla} \rho^\gamma = \boldsymbol{f}. \tag{1.1b}$$

The quantities $\rho \geq 0$ and $\boldsymbol{u} = (u_1, .., u_d)^T$ are respectively the density and velocity of the fluid, while $\boldsymbol{f}$ is an external force. The pressure satisfies the ideal gas law with $a > 0$ and $\gamma > 1$. Equation (1.1a) expresses the local conservation of the mass of the fluid while equation (1.1b) expresses the local balance between momentum and forces. The viscosity coefficients μ and λ are such that $\mu > 0$ and $\mu + \lambda > 0$. System (1.1) is complemented with homogeneous Dirichlet boundary conditions on the velocity:

$$\boldsymbol{u}|_{\partial\Omega} = 0, \tag{1.2}$$

and the following average density constraint (up to the normalization by $|\Omega|$ it is the same as prescribing the total mass)

$$\frac{1}{|\Omega|} \int_\Omega \rho \, \mathrm{d}\boldsymbol{x} = \rho^\star > 0. \tag{1.3}$$

Regarding the theoretical results on these equations, the existence of weak solutions has been first proved by Lions in [33] for adiabatic exponents $\gamma > \frac{5}{3}$ in dimension $d = 3$, a result which has then been extended to coefficients $\gamma \in (\frac{3}{2}, \frac{5}{3})$ by Novo and Novotný in [34]. It is also worth mentioning that a recent paper of Plotnikov and Weigant [36] improves these previous results, but this improvement will be not treated in this paper.

From the numerical viewpoint, compressible fluid equations have been intensively studied and several approximations have been designed in the last few years. In this paper, we consider a stabilized (and stationary) version of a numerical scheme, a linearized version of which (following pressure correction techniques) is implemented in the industrial software CALIF^{3}S [3] developed by the french *Institut de Radioprotection et de Sûreté Nucléaire* (IRSN, a research center devoted to nuclear safety). This scheme falls in the class of *staggered* discretizations in the sense that the scalar variables (density, pressure) are associated with the cells of a primal mesh $\mathcal{M}$ while the vectorial variables (velocity, external force) are associated with the set $\mathcal{E}$ of faces of the primal mesh. Such decoupling, associated here with a Crouzeix-Raviart finite element discretization [5] (but other non-conforming finite elements are possible, such as the Rannacher-Turek discretization [37]) of the viscous stress tensor, provides a discrete pressure estimate, thanks to the so-called discrete *inf-sup* stability condition (see for instance [24]). This condition, which is also satisfied by the MAC scheme (see [25], [26], [27]) on structured grids, ensures the unconditional stability of the scheme in almost incompressible regimes (for instance in the low Mach regime, see [19] and [28]). Let us mention that, contrary to the MAC scheme (where the domain Ω is assumed to be a finite union of orthogonal parallelepipeds, and the mesh is composed by a structured partition of rectangular parallelepipeds with cell faces normal to the coordinate axis), the scheme considered in this paper is able to cope with unstructured meshes.

In its reduced form, our numerical scheme reads

$$\mathrm{div}_{\mathcal{M}}(\rho \boldsymbol{u}) + T^1_{\mathrm{stab}} + T^2_{\mathrm{stab}} = 0,$$

$$\mathbf{div}_{\mathcal{E}}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \boldsymbol{\Delta}_{\mathcal{E}} \boldsymbol{u} - (\mu + \lambda)(\boldsymbol{\nabla} \circ \mathrm{div})_{\mathcal{E}} \boldsymbol{u} + a \boldsymbol{\nabla}_{\mathcal{E}}(\rho^\gamma) + T^3_{\mathrm{stab}} = \widetilde{\Pi}_{\mathcal{E}} \boldsymbol{f},$$

where, as suggested by the notations used for the discrete differential operators, the (scalar) mass equation is discretized on the primal mesh $\mathcal{M}$, whereas the (vectorial) momentum equation is discretized on a dual mesh associated with the set of faces $\mathcal{E}$.

The finite element discretization for the viscous stress tensor is here coupled with finite volume discretizations of the convective terms which allow, thanks to standard techniques, to obtain discrete convection operators satisfying maximum principles (*e.g.* [30]). The discrete mass convection operator is a standard finite volume operator defined on the cells of the primal mesh $\mathcal{M}$ while the discrete momentum convection operator is also a finite volume operator written on *dual cells*, *i.e.* cells centered at the location of the velocity unknowns, namely the faces $\mathcal{E}$. A difficulty implied by such staggered discretization lies in the fact that, as in the continuous case, the derivation of the energy inequality needs that a mass balance equation be satisfied on the same (dual) cells, while the mass balance in the scheme is naturally written on the primal cells. A procedure has therefore been developed to define the density on the dual mesh cells and the mass fluxes through the dual faces from the primal cell density and the primal faces mass fluxes, which ensures a discrete mass balance on dual cells.

Compared to the continuous problem (1.1), the discrete equations contain three additional "stabilization" terms T^i_{stab} that ensure the convergence (up to extracting a subsequence) of the numerical solutions towards weak solutions of (1.1)-(1.2)-(1.3) as the mesh size tends to 0. The first stabilization term T^1_{stab} guarantees the total mass constraint (1.3) at the discrete level. The second stabilization term T^2_{stab}, which is a discrete counterpart of a diffusion term for the density, provides an additional (mesh dependent) estimate on the discrete gradient of the density. As we will explain in details in the core of the paper, this artificial discrete diffusion is used to show the crucial convergence property satisfied by the effective viscous flux. The last stabilization term T^3_{stab} is an artificial pressure gradient which is necessary only if $\gamma \leq 3$. The precise definitions of the discrete operators and stabilization terms are given in Section 3.

There exist in the literature several recent convergence results for finite element or mixed finite volume - finite element schemes. In [13], Eymard *et al.* (see also [20] for the particular case $\gamma = 1$, *i.e.* a linear pressure term) study the compressible Stokes equations, that correspond to (1.1) where the nonlinear convective term $\text{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u})$ is neglected. At the discrete level, two stabilization terms, namely, T^1_{stab} and a term similar to T^2_{stab}, are introduced for the convergence analysis of the numerical scheme. In the case of Equations (1.1)-(1.2)-(1.3), *i.e.* with the additional convective term, Gallouët *et al.* prove in the recent paper [22] the convergence of the MAC scheme under the condition $\gamma > 3$, with only one stabilization term T^1_{stab} ensuring the mass constraint (1.3) (we refer to Remark 5.1 below which explains why T^2_{stab} is unnecessary for the MAC scheme). Finally, Karper proved in [29] (see also the recent book [15]) a convergence result in the evolution case, again for $\gamma > 3$, and an equivalent of the artificial diffusion term T^2_{stab} is also introduced (note that in the evolutionary case there is no additional mass constraint and thus no need for T^1_{stab}). Let us mention that for the evolutionary case, error estimates are available in [23] for the whole range $\gamma > \frac{3}{2}$. For values of $\gamma \leq \frac{3}{2}$ (more precisely $\gamma < 2$), convergence results can be found in [16] and [17] within the framework of dissipative measure-valued solutions, a "weaker" framework than ours.

To the best of our knowledge, our result is the first convergence result in the three-dimensional case for values $\gamma \in (\frac{3}{2}, 3]$ within the framework of weak solutions with finite energy (see Definition 2.1). It provides an alternative proof of the existence result obtained by Lions or by Novo and Novotný. Compared to the previous numerical studies dealing with coefficients $\gamma > 3$, it requires the introduction of a third stabilization term T^3_{stab}, an artificial pressure term weighted by some power of the mesh size: $h^\xi \boldsymbol{\nabla}_\mathcal{E}(\rho^\Gamma)$ with $\Gamma > 3$. Note that the stabilization terms T^2_{stab} and T^3_{stab} are not implemented in practice and are introduced here for the convergence analysis.

Let us emphasize that the evolution case is beyond the scope of this paper and left for future work.

The paper is organized as follows: in Section 2, we present the main ingredients for the analysis of the continuous problem (1.1)-(1.2)-(1.3). This section does not present any substantial novelty compared to the work of Novo and Novotnỳ [34], and the reader already familiar with the analysis of the compressible Navier-Stokes equations can directly pass to the next sections concerning the discrete problem. Then, in Section 3, we introduce our numerical scheme and state precisely our main convergence result. We derive in Section 4 mesh independent estimates and show the existence of solutions to the numerical scheme. Finally, Section 5 is devoted to the proof of convergence of the numerical method as the mesh size tends to 0. We provide in the Appendix additional material and proofs.

2 The continuous setting

The aim of this section is to present the main ingredients involved in the analysis of the continuous problem (1.1)-(1.2)-(1.3) for readers who are not familiar with compressible Navier-Stokes equations. Although the existence theory of weak solutions to these equations is now well understood since the works of Lions [33] and Feireisl [14] (see also [35]), the analysis developed there involves advanced tools (such as weak compactness methods based on energy estimates, renormalized solutions, effective viscous flux, etc.) that are to our opinion worth recalling. Especially as these tools will be also crucial in the convergence analysis of our numerical scheme.

It turns out that the estimates and compactness arguments differ significantly according to the value of the adiabatic exponent γ appearing in the pressure law. For $d = 3$ and $\gamma > 3$, the case treated in previous numerical studies, a sketch of the proof of the stability of weak solutions can be found for instance in [22]. We focus here, as in the other sections, on the case $d = 3$ and $\gamma \in (\frac{3}{2}, 3]$ which is the case covered by the study of Novo and Novotný.

Essentially, the minimal value $\gamma^* = 3$ is the one that ensures a control of the pressure ρ^γ and of the convective term $\rho \boldsymbol{u} \otimes \boldsymbol{u}$ in $\mathrm{L}^2(\Omega)$. The value $\gamma^* = \frac{5}{3}$ exhibited by Lions corresponds to the minimal exponent guaranteeing that ρ is controlled in $\mathrm{L}^2(\Omega)$. As we will explain later on (see Remark 2.4 below), this control is required to prove that weak solutions are renormalized solutions. This constraint on γ has been relaxed by Novo, Novotný [34] (and Feireisl [14] in the evolutionary case) to reach $\gamma > \frac{3}{2}$ which corresponds to the minimal exponent ensuring that $\rho \boldsymbol{u} \otimes \boldsymbol{u}$ is controlled in $\mathrm{L}^p(\Omega)$, with $p > 1$. The interested reader is also referred to the paper of Plotnikov and Weigant [36] for the case $1 < \gamma \leq \frac{3}{2}$, case which will not be treated in the present paper.

As said before, this section does not present any substantial novelty compared to Novo and Novotný's work and the reader already familiar with the analysis of compressible Navier-Stokes equations can directly pass to the next sections concerning the discrete problem. Note however that for numerical purposes, we present below an original treatment of the convective term in the analysis of the effective viscous flux (see Subsection 2.4.1). This alternative method allows to circumvent the use of abstract tools for the compensated compactness theory, namely the famous Div-Curl Lemma and a Commutator Lemma (see the method employed by Novotný and Straskrăba in [35] Section 4.4). These tools are indeed a little bit cumbersome to adapt in the discrete setting (although this has been achieved by Karper in [29, 15] for the evolutionary case), which motivates our alternative method based on a regularization of the velocity.

This section is organized as follows: after recalling the classical definition of weak solutions to problem (1.1)-(1.2)-(1.3), we derive a priori estimates and show the stability of weak solutions. In the last subsection we explain how to approximate (1.1) in order to construct effectively weak solutions.

2.1 Definition of weak solutions, stability result

Definition 2.1. *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$. Let $\gamma > \frac{3}{2}$. Let $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ and $\rho^\star > 0$. A pair $(\rho, \boldsymbol{u}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}^1_0(\Omega)$ is said to be a weak solution to Problem (1.1)-(1.2)-(1.3) if it satisfies:*

Positivity of the density and global mass constraint:

$$\rho \geq 0 \text{ a.e. in } \Omega \quad \text{and} \qquad \frac{1}{|\Omega|}\int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = \rho^\star. \tag{2.1}$$

Equations (1.1a)–(1.1b) are satisfied in the weak sense:

$$\int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \phi \,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall \phi \in \mathcal{C}^\infty_c(\Omega), \tag{2.2}$$

$$- \int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - a \int_\Omega \rho^\gamma \operatorname{div} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \\ + (\lambda + \mu) \int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{v} \in \mathcal{C}^\infty_c(\Omega)^3. \tag{2.3}$$

The pair $(\rho, \boldsymbol{u}) \in L^{3(\gamma-1)}(\Omega) \times \mathbf{H}^1_0(\Omega)$ *is said to be a weak solution with bounded energy if, in addition to the previous conditions, it satisfies the energy inequality*

$$\mu \int_\Omega |\boldsymbol{\nabla} \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} + (\lambda + \mu) \int_\Omega (\operatorname{div} \boldsymbol{u})^2 \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} \tag{2.4}$$

Finally the pair $(\rho, \boldsymbol{u}) \in L^{3(\gamma-1)}(\Omega) \times \mathbf{H}^1_0(\Omega)$ *is said to be a weak renormalized solution if, in addition to the previous conditions and for any* $b \in \mathcal{C}^0([0, +\infty)) \cap \mathcal{C}^1((0, +\infty))$ *such that*

$$|b'(t)| \leq \begin{cases} ct^{-\lambda_0}, \ \lambda_0 < 1 & \text{if } t < 1, \\ ct^{\lambda_1}, \ \lambda_1 + 1 \leq \frac{3(\gamma-1)}{2} & \text{if } t \geq 1, \end{cases} \tag{2.5}$$

the pair $(\rho, \boldsymbol{u})$ *satisfies*

$$\operatorname{div}(b(\rho)\boldsymbol{u}) + \big(b'(\rho)\rho - b(\rho)\big) \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3), \tag{2.6}$$

where ρ *and* $\boldsymbol{u}$ *have been extended by* 0 *outside* Ω*.*

Remark 2.1. *In the whole paper, we adopt the following notations:*

$$\mathbf{H}^1_0(\Omega) := \mathrm{H}^1_0(\Omega)^d, \qquad \mathbf{W}^{1,p}_0(\Omega) := \mathrm{W}^{1,p}_0(\Omega)^d, \qquad \mathbf{L}^p(\Omega) := \mathrm{L}^p(\Omega)^d, \ p \in [1, +\infty].$$

Remark 2.2. *Since* $\gamma > \frac{3}{2}$*, we have* $\rho\boldsymbol{u} \in \mathbf{L}^{\frac{6}{5}}(\Omega)$ *and by density (2.2) is valid for all* $\phi \in \mathrm{W}^{1,6}_0(\Omega)$*. In addition,* $\rho\boldsymbol{u} \otimes \boldsymbol{u} \in \mathbf{L}^{1+\eta}(\Omega)^3$ *and* $\rho^\gamma \in \mathrm{L}^{1+\eta}(\Omega)$ *for some* $\eta > 0$ *so that (2.3) is valid for all* $\boldsymbol{v} \in \mathbf{W}^{1,q}_0(\Omega)$ *for all* $q \in [1, +\infty)$*.*

Remark 2.3. *For* $d = 3$*,* $\gamma > 3$*, and* $d = 2$*,* $\gamma > 1$*, we would get better integrability on* ρ*. Precisely, we would have* $\rho \in \mathrm{L}^{2\gamma}(\Omega)$*.*

Remark 2.4.

- *When* γ *is large enough, namely* $\gamma \geq \frac{5}{3}$*, the following lemma, initially proved by Di Perna and Lions [7], shows that any weak solution with finite energy of (1.1) is a renormalized weak solution.*

  Lemma 2.1 ([35] Lemma 3.3)**.** *Assume that* $\gamma \geq \frac{5}{3}$ *and let* $\rho \in \mathrm{L}^{3(\gamma-1)}_{\mathrm{loc}}(\mathbb{R}^3)$*,* $\boldsymbol{u} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ *satisfying the continuity equation*

  $$\operatorname{div}(\rho\boldsymbol{u}) = 0 \quad \text{in } \ \mathcal{D}'(\mathbb{R}^3).$$

  Then, equation (2.6) holds for any $b \in \mathcal{C}^0([0, +\infty)) \cap \mathcal{C}^1((0, +\infty))$ *satisfying (2.5).*

  More precisely, the justification of the renormalized equation requires a preliminary regularization of the density. The commutator term resulting from this regularization involves in particular products like $\rho \operatorname{div} \boldsymbol{u}$ *which are then controlled precisely under the condition that* $\rho \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ *since* $\operatorname{div} \boldsymbol{u} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ *(see for instance [35] Lemma 3.1). This condition is achieved as soon as* $3(\gamma - 1) \geq 2$*, i.e.* $\gamma \geq \frac{5}{3}$*. The interested reader is also referred to [18] (Appendix B) for a discussion on the criticality of the assumption on* ρ*.*

- *If the pair $(\rho, \boldsymbol{u}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}_0^1(\Omega)$ is a renormalized solution which satisfies, instead of the continuity equation (2.2),*

$$\operatorname{div}(\rho \boldsymbol{u}) = g \quad \text{in } \mathcal{D}'(\Omega) \quad \text{for some } g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3),$$

then, extending ρ, $\boldsymbol{u}$ by zero outside Ω (denoting again $\rho, \boldsymbol{u}, g$ the extended functions), the previous equation also holds in $\mathcal{D}'(\mathbb{R}^3)$. Moreover, for any $b \in \mathcal{C}^1([0,+\infty))$ satisfying (2.5), denoting b_M the truncated function such that

$$b_M(t) = \begin{cases} b(t) & \text{if } t < M, \\ b(M) & \text{if } t \geq M, \end{cases}$$

then we have

$$\operatorname{div}(b_M(\rho)\boldsymbol{u}) + \big([b_M]'_+(\rho)\rho - b_M(\rho)\big)\operatorname{div}\boldsymbol{u} = g\,[b_M]'_+(\rho) \quad \text{in } \mathcal{D}'(\mathbb{R}^3) \tag{2.7}$$

where

$$[b_M]'_+(t) = \begin{cases} b'(t) & \text{if } t < M, \\ 0 & \text{if } t \geq M. \end{cases}$$

We now focus on the stability of weak solutions the proof of which is essential for the analysis of the numerical scheme in the next sections. In Section 2.5, some elements are given for the approximation procedure that allows to construct such weak solutions.

Theorem 2.2. *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$. Assume that $\gamma \in (\frac{3}{2}, 3]$. Consider sequences of external forces $(\boldsymbol{f}_n)_{n\in\mathbb{N}} \subset \mathbf{L}^2(\Omega)$ and masses $(\rho_n^\star)_{n\in\mathbb{N}} \subset \mathbb{R}_+^*$, and an associated sequence $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ of renormalized weak solutions with bounded energy. Assume that $\rho_n^\star \to \rho^\star > 0$ and that $(\boldsymbol{f}_n)_{n\in\mathbb{N}}$ converges strongly in $\mathbf{L}^2(\Omega)$ to $\boldsymbol{f}$. Then, there exist $(\rho, \boldsymbol{u}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}_0^1(\Omega)$ and a subsequence of $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$, still denoted $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ such that:*

- *The sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{u}$ in $\mathbf{L}^q(\Omega)$ for all $q \in [1, 6)$,*
- *The sequence $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ in $\mathrm{L}^q(\Omega)$ for all $q \in [1, 3(\gamma-1))$ and weakly in $\mathrm{L}^{3(\gamma-1)}(\Omega)$,*
- *The sequence $(\rho_n^\gamma)_{n\in\mathbb{N}}$ converges to ρ^γ in $\mathrm{L}^q(\Omega)$ for all $q \in [1, \frac{3(\gamma-1)}{\gamma})$ and weakly in $\mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$,*
- *The pair $(\rho, \boldsymbol{u})$ is a weak solution of Problem (1.1)-(1.2)-(1.3) with finite energy.*

The proof of Theorem 2.2 is divided into four steps: first we derive the basic uniform estimates which enable us in the next step to derive compactness results on the sequence $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ and to pass to the limit in the mass and momentum equations as $n \to +\infty$. Then, we prove some weak compactness property on the "effective viscous flux" which eventually allows us to prove the strong convergence of the density and to pass to the limit in the equation of state (*i.e.* in the pressure law).

2.2 Uniform estimates

Proposition 2.3 (Control of the velocity)**.** *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$. Assume that $\gamma \in (\frac{3}{2}, 3]$. Let $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ be the sequence defined in Theorem 2.2. Then, we have the following* a priori *control on the velocity:*

$$\|\boldsymbol{u}_n\|_{\mathbf{H}^1_0(\Omega)} \leq C(\Omega, (\|\boldsymbol{f}_n\|_{\mathbf{L}^2(\Omega)})_{n\in\mathbb{N}}) \leq C, \qquad \forall n \in \mathbb{N}. \tag{2.8}$$

Proof. The result directly follows from the energy inequality (2.4), the Poincaré inequality and Young's inequality. □

An additional estimate has to be derived to get a control on the pressure (and thus on the density). To that end, we define for $q \in (1, +\infty)$

$$\mathrm{L}^q_0(\Omega) = \big\{p \in \mathrm{L}^q(\Omega),\ \text{s.t.}\ < p >= \frac{1}{|\Omega|}\int_\Omega p \,\mathrm{d}\boldsymbol{x} = 0\big\}$$

and we recall the following result.

Lemma 2.4. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^d$, $d \geq 1$. Then, there exists a linear operator $\mathcal{B}$ depending only on Ω with the following properties:*

(i) For all $q \in (1, +\infty)$,

$$\mathcal{B} : \mathrm{L}^q_0(\Omega) \to \mathbf{W}^{1,q}_0(\Omega).$$

(ii) For all $q \in (1, +\infty)$ and $p \in \mathrm{L}^q_0(\Omega)$,

$$\mathrm{div}(\mathcal{B}p) = p, \ \textit{a.e. in}\ \Omega.$$

(iii) For all $q \in (1, +\infty)$, there exists $C = C(q, \Omega)$, such that for any $p \in \mathrm{L}^q_0(\Omega)$:

$$|\mathcal{B}p|_{\mathbf{W}^{1,q}(\Omega)} \leq C\, \|p\|_{\mathrm{L}^q(\Omega)}.$$

Operator $\mathcal{B}$ is the so-called Bogovskii operator. The interested reader is referred to [35] (Chapter 3.3) for a proof and additional properties on this operator. In particular, the operator $\mathcal{B}$ is independent of q.

Proposition 2.5 (Control of the density and pressure)**.** *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$. Assume that $\gamma \in (\frac{3}{2}, 3]$. Let $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ be the sequence defined in Theorem 2.2. Then, we have the following* a priori *estimate on the density: there exists a constant C such that:*

$$\|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)} \leq C, \qquad \forall n \in \mathbb{N}. \tag{2.9}$$

As a consequence, the pressure $(\rho_n^\gamma)_{n\in\mathbb{N}}$ is controlled in $\mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$.

Proof. Let us set (recall that we focus here on the case $\gamma \in (\frac{3}{2}, 3]$)

$$\eta = \frac{2\gamma - 3}{\gamma} \in (0, 1].$$

Observe that $\gamma(1+\eta) = 3(\gamma - 1)$. Let $n \in \mathbb{N}$ and define $P_n = \rho_n^\gamma$. Applying Lemma 2.4 to $P_n - < P_n >$ and using the resulting field $\boldsymbol{v}_n = \mathcal{B}(P_n - < P_n >)$ as a test function in (2.3), one gets

$$\begin{aligned} a \int_\Omega (\rho_n^\gamma)^{1+\eta} \, \mathrm{d}\boldsymbol{x} &= \frac{a}{|\Omega|} \int_\Omega \rho_n^\gamma \int_\Omega (\rho_n^\gamma)^\eta \, \mathrm{d}\boldsymbol{x} - \int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} + \mu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_n : \boldsymbol{\nabla} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} \\ &\quad + (\lambda + \mu) \int_\Omega \operatorname{div} \boldsymbol{u}_n \operatorname{div} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}_n \cdot \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} \\ &= I_1 + \cdots + I_5. \end{aligned} \tag{2.10}$$

Before estimating the various integrals of the right-hand side, note that from Lemma 2.4

$$\begin{aligned} \|\boldsymbol{\nabla} \boldsymbol{v}_n\|_{\mathbf{L}^q(\Omega)^3} &\le C \|\rho_n^{\gamma\eta} - < \rho_n^{\gamma\eta} >\|_{\mathrm{L}^q(\Omega)} \\ &\le C \|(\rho_n^\gamma)^{q\eta}\|_{\mathrm{L}^1(\Omega)}^{\frac{1}{q}} + C \|(\rho_n^\gamma)^\eta\|_{\mathrm{L}^1(\Omega)} \\ &\le C \|\rho_n^\gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^\eta \end{aligned}$$

provided that

$$1 \le q \le \frac{1+\eta}{\eta} = \frac{3(\gamma - 1)}{2\gamma - 3}.$$

In particular, since $\gamma \in (\frac{3}{2}, 3]$, we have

$$\frac{3(\gamma-1)}{2\gamma-3} \ge 2 \quad \text{and thus} \quad \|\boldsymbol{\nabla} \boldsymbol{v}_n\|_{\mathbf{L}^2(\Omega)^3} \le C \|\rho_n^\gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^\eta.$$

We use this control to estimate the integrals I_3, I_4 and I_5:

$$\begin{aligned} |I_3 + I_4 + I_5| &\le C \big(\|\boldsymbol{\nabla} \boldsymbol{u}_n\|_{\mathbf{L}^2(\Omega)^3} + \|\boldsymbol{f}_n\|_{\mathbf{L}^2(\Omega)} \big) |\boldsymbol{v}_n|_{\mathbf{W}^{1,2}(\Omega)} \\ &\le C \big(\|\boldsymbol{\nabla} \boldsymbol{u}_n\|_{\mathbf{L}^2(\Omega)^3} + \|\boldsymbol{f}_n\|_{\mathbf{L}^2(\Omega)} \big) \|\rho_n^\gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^\eta \\ &\le C + \frac{a}{4} \|\rho_n^\gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{1+\eta} \end{aligned}$$

using Young's inequality and the control of the velocity (2.8). It remains to estimate the integrals I_1 and I_2. We first have

$$|I_1| \le C(a, \Omega) \left(\int_\Omega \rho_n^\gamma \, \mathrm{d}\boldsymbol{x} \right) \|\rho_n^\gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^\eta \le C(a, \Omega) \|\rho_n\|_{\mathrm{L}^\gamma(\Omega)}^\gamma \|\rho_n^\gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^\eta$$

and by an interpolation inequality

$$\|\rho_n\|_{\mathrm{L}^\gamma(\Omega)}^\gamma \le \|\rho_n\|_{\mathrm{L}^1(\Omega)}^{\gamma(1-r)} \|\rho_n\|_{\mathrm{L}^{\gamma(1+\eta)}(\Omega)}^{\gamma r} \quad \text{with} \quad \frac{1}{\gamma} = \frac{r}{\gamma(1+\eta)} + (1-r)$$

that is

$$r = \frac{(\gamma-1)(1+\eta)}{\gamma(1+\eta) - 1} \in (0, 1) \quad \text{for} \quad \eta = \frac{2\gamma - 3}{\gamma}.$$

Since $r < 1$, we can use Young's inequality and the control of the mass (1.3) to deduce that

$$|I_1| \le C \|\rho_n^\gamma\|^r_{\mathrm{L}^{1+\eta}(\Omega)} \|\rho_n^\gamma\|^\eta_{\mathrm{L}^{1+\eta}(\Omega)} \le \frac{a}{4} \|\rho_n^\gamma\|^{1+\eta}_{\mathrm{L}^{1+\eta}(\Omega)} + C.$$

Finally, the integral of the convective term is controlled as follows:

$$\begin{aligned}|I_2| &\le C \|\rho_n\|_{\mathrm{L}^{\gamma(1+\eta)}(\Omega)} \|\boldsymbol{u}_n\|^2_{\mathbf{L}^6(\Omega)} \|\boldsymbol{\nabla v}_n\|_{\mathbf{L}^{\frac{1+\eta}{\eta}}(\Omega)^3} \\ &\le C \|\rho_n^\gamma\|^{1/\gamma}_{\mathrm{L}^{1+\eta}(\Omega)} \|\boldsymbol{u}_n\|^2_{\mathbf{L}^6(\Omega)} \|\rho_n^\gamma\|^\eta_{\mathrm{L}^{1+\eta}(\Omega)} \\ &\le C \|\boldsymbol{u}_n\|^2_{\mathbf{L}^6(\Omega)} \|\rho_n^\gamma\|^{\eta+\frac{1}{\gamma}}_{\mathrm{L}^{1+\eta}(\Omega)}\end{aligned}$$

so that, by the control of the velocity and by Young's inequality ($\gamma > 1$) :

$$|I_2| \le C + \frac{a}{4} \|\rho_n^\gamma\|^{1+\eta}_{\mathrm{L}^{1+\eta}(\Omega)}.$$

Coming back to (2.10) and gathering all the previous estimates, we have

$$a \int_\Omega (\rho_n^\gamma)^{1+\eta} \, \mathrm{d}\boldsymbol{x} \le C + \frac{3}{4} a \int_\Omega (\rho_n^\gamma)^{1+\eta} \, \mathrm{d}\boldsymbol{x}.$$

As a consequence, we deduce the control of the density and the pressure (we recall that $\eta = \frac{2\gamma-3}{\gamma}$):

$$\|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)} + \|\rho_n^\gamma\|_{\mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)} \le C.$$

□

2.3 Passing to the limit in the mass and momentum equations

Thanks to the previously derived estimates, we have the following result.

Proposition 2.6. *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$. Assume that $\gamma \in (\frac{3}{2}, 3]$. Let $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ be the sequence defined in Theorem 2.2. There exist $(\rho, \boldsymbol{u}, \overline{\rho^\gamma}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}^1_0(\Omega) \times \mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$ such that up to extraction of a subsequence, the following convergences hold as $n \to +\infty$:*

$$\begin{aligned}&\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{weakly in } \mathbf{H}^1_0(\Omega) \text{ and strongly in } \mathbf{L}^q(\Omega),\ \forall q \in [1, 6), \\ &\rho_n \rightharpoonup \rho \quad \text{weakly in } \mathrm{L}^{3(\gamma-1)}(\Omega), \\ &\rho_n^\gamma \rightharpoonup \overline{\rho^\gamma} \quad \text{weakly in } \mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega).\end{aligned}$$

Combining the weak convergence of the density and the strong convergence of the velocity we have

$$\rho_n \boldsymbol{u}_n \rightharpoonup \rho \boldsymbol{u} \quad \text{weakly in } \mathbf{L}^q(\Omega), \quad \forall q \in [1, \tfrac{6(\gamma-1)}{\gamma+1}), \tag{2.11}$$

$$\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n \rightharpoonup \rho \boldsymbol{u} \otimes \boldsymbol{u} \quad \text{weakly in } \mathbf{L}^q(\Omega)^3, \quad \forall q \in [1, \tfrac{3(\gamma-1)}{\gamma}). \tag{2.12}$$

Passing to the limit $n \to +\infty$ in the mass constraint and in in the weak formulation of the mass and momentum equations, the triplet $(\rho, \mathbf{u}, \overline{\rho^\gamma})$ is seen to satisfy:

Positivity of the density and global mass constraint:

$$\rho \geq 0 \ \textit{a.e. in} \ \ \Omega \quad \textit{and} \qquad \frac{1}{|\Omega|}\int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = \rho^\star. \tag{2.13}$$

Continuity and the momentum equations in the weak sense:

$$\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\phi \,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall \phi \in \mathcal{C}_c^\infty(\Omega), \tag{2.14}$$

$$-\int_\Omega \rho\boldsymbol{u}\otimes\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - a\int_\Omega \overline{\rho^\gamma}\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \mu\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ + (\lambda+\mu)\int_\Omega \mathrm{div}\,\boldsymbol{u}\ \mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{v} \in \mathcal{C}_c^\infty(\Omega)^3. \tag{2.15}$$

Moreover the energy inequality is satisfied at the limit:

$$\mu\int_\Omega |\boldsymbol{\nabla}\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} + (\lambda+\mu)\int_\Omega (\mathrm{div}\,\boldsymbol{u})^2\,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}. \tag{2.16}$$

Remark 2.5. *For $\gamma > \frac{3}{2}$ we guarantee that $\frac{3(\gamma-1)}{\gamma} > 1$ and thus the convective term in (2.15) is such that*

$$\rho\boldsymbol{u}\otimes\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v} \in \mathrm{L}^r(\Omega) \quad \textit{for some} \quad r > 1 \quad \forall\ \boldsymbol{v} \in \mathbf{W}_0^{1,\infty}(\Omega).$$

To complete the proof of Theorem 2.2, it remains to identify the limit pressure $\overline{\rho^\gamma}$ in (2.15), that is to pass to the limit in the equation of state and prove that

$$\overline{\rho^\gamma} = \rho^\gamma \quad \text{a.e. in } \Omega \tag{2.17}$$

which is equivalent to proving the strong convergence of the density towards its weak limit.

2.4 Passing to the limit in the equation of state

This is classically obtained in two steps: first by proving some weak compactness property satisfied by the so-called effective viscous flux defined as $(2\mu+\lambda)\mathrm{div}\,\boldsymbol{u} - a\rho^\gamma$, and then, by using the monotonicity of the pressure to deduce the strong convergence of the sequence of densities $(\rho_n)_{n\in\mathbb{N}}$.

2.4.1 Weak compactness of the effective viscous flux

Let us first recall the definition of the curl operator, and a useful identity linked to this operator.

Lemma 2.7 (A differential identity)**.** *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$. For $\boldsymbol{a} = (a_1,a_2,a_3)^T$ and $\boldsymbol{b} = (b_1,b_2,b_3)^T$ in $\mathbb{R}^3$ we denote $\boldsymbol{a}\wedge\boldsymbol{b} = (a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1)^T \in \mathbb{R}^3$. For a vector valued function $\boldsymbol{v} = (v_1,v_2,v_3)^T$, denote $\mathrm{curl}\,\boldsymbol{v} = \boldsymbol{\nabla}\wedge\boldsymbol{v}$ where $\boldsymbol{\nabla} = (\partial_1,\partial_2,\partial_3)^T$. With these notations, if $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ and $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, the following identity holds:*

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{curl}\,\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} \\ + \int_{\partial\Omega} (\boldsymbol{\nabla}\boldsymbol{v}.\boldsymbol{n})\cdot\boldsymbol{u}\ \mathrm{d}\sigma(\boldsymbol{x}) + \int_{\partial\Omega} \mathrm{curl}\,\boldsymbol{v}\cdot(\boldsymbol{u}\wedge\boldsymbol{n})\ \mathrm{d}\sigma(\boldsymbol{x}) - \int_{\partial\Omega} \mathrm{div}\,\boldsymbol{v}\,(\boldsymbol{u}\cdot\boldsymbol{n})\ \mathrm{d}\sigma(\boldsymbol{x}). \tag{2.18}$$

If $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$ *and* $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$*, this identity simplifies to:*

$$\int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}. \tag{2.19}$$

We shall also need the next result.

Lemma 2.8. *Let* Ω *be a bounded open set of* $\mathbb{R}^d$*. Then, there exists a linear operator* $\mathcal{A}$ *with the following properties:*

(i) For all $q \in (1, +\infty)$*,*

$$\mathcal{A} : \mathrm{L}^q(\Omega) \to \mathbf{W}^{1,q}(\Omega).$$

(ii) For all $q \in (1, +\infty)$ *and* $p \in \mathrm{L}^q(\Omega)$*,*

$$\operatorname{div}(\mathcal{A}p) = p, \ \textit{and} \ \operatorname{curl}(\mathcal{A}p) = 0, \ \textit{a.e. in } \Omega.$$

(iii) For all $q \in (1, +\infty)$*, there exists* $C = C(q, \Omega)$*, such that for any* $p \in \mathrm{L}^q(\Omega)$*:*

$$|\mathcal{A}p|_{\mathbf{W}^{1,q}(\Omega)} \le C \, \|p\|_{\mathrm{L}^q(\Omega)}.$$

Proof. A solution is given by $\mathcal{A}p := \boldsymbol{\nabla}\Delta^{-1}(p)$, where Δ^{-1} is defined as the inverse of the Laplacian on $\mathbb{R}^3$, here applied to p extended by 0 outside Ω. The reader is referred to [35] Section 4.4.1 for properties of the operator $\mathcal{A}$. In particular the operator $\mathcal{A}$ does not depend on q. □

Proposition 2.9. *Let* Ω *be a Lipschitz bounded domain of* $\mathbb{R}^3$*. Assume that* $\gamma \in (\frac{3}{2}, 3]$*. Let* $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ *be the sequence defined in Theorem 2.2. For* $k \in \mathbb{N}^*$*, define*

$$T_k(t) = \begin{cases} t & \textit{if } t \in [0, k), \\ k & \textit{if } t \in [k, +\infty). \end{cases} \tag{2.20}$$

The sequence $(T_k(\rho_n))_{n\in\mathbb{N}}$ *is bounded in* $\mathrm{L}^\infty(\Omega)$ *and up to extracting a subsequence, it converges for the weak-* topology in* $\mathrm{L}^\infty(\Omega)$ *towards some function denoted* $\overline{T_k(\rho)}$*. Then the following identity holds:*

$$\lim_{n\to+\infty} \int_\Omega \big((2\mu+\lambda) \operatorname{div} \boldsymbol{u}_n - a\rho_n^\gamma\big) T_k(\rho_n) \phi \, \mathrm{d}\boldsymbol{x} = \int_\Omega \big((2\mu+\lambda) \operatorname{div} \boldsymbol{u} - a\overline{\rho^\gamma}\big) \overline{T_k(\rho)} \phi \, \mathrm{d}\boldsymbol{x}, \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega). \tag{2.21}$$

Proof of Proposition 2.9. Let $k \in \mathbb{N}^*$. For $n \in \mathbb{N}$, let $\boldsymbol{w}_n = \mathcal{A}T_k(\rho_n)$ be the field associated with $T_k(\rho_n)$ through Lemma 2.8. We have

$$\operatorname{div} \boldsymbol{w}_n = T_k(\rho_n), \quad \operatorname{curl} \boldsymbol{w}_n = 0, \quad (\boldsymbol{w}_n)_{n\in\mathbb{N}} \text{ is bounded in } \mathbf{W}^{1,q}(\Omega) \quad \forall q \in (1, +\infty).$$

Moreover, $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^\infty(\Omega)$ and up to extracting a subsequence, as $n \to +\infty$, it strongly converges in $\mathbf{L}^q(\Omega)$ and weakly in $\mathbf{W}^{1,q}(\Omega)$ for all $q \in (1, +\infty)$ towards some function $\boldsymbol{w}$ satisfying:

$$\operatorname{div} \boldsymbol{w} = \overline{T_k(\rho)} \quad \text{and} \quad \operatorname{curl} \boldsymbol{w} = 0. \tag{2.22}$$

Let $\phi \in \mathcal{C}_c^\infty(\Omega)$. Considering in (2.3) the test function $\boldsymbol{v}_n = \phi \boldsymbol{w}_n$ we get:

$$- \int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} - a \int_\Omega \rho_n^\gamma \operatorname{div}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} \\ + (\lambda + \mu) \int_\Omega \operatorname{div} \boldsymbol{u}_n \operatorname{div}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot (\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x}.$$

Using the formula (2.19) and the fact that $\operatorname{div}(\phi \boldsymbol{w}_n) = T_k(\rho_n)\phi + \boldsymbol{w}_n \cdot \boldsymbol{\nabla}\phi$ and $\operatorname{curl}(\phi \boldsymbol{w}_n) = L(\phi)\boldsymbol{w}_n$ where $L(\phi)$ is a matrix involving first order derivatives of ϕ, we obtain:

$$\begin{aligned}
&\int_\Omega \big(a\rho_n^\gamma - (2\mu + \lambda)\operatorname{div} \boldsymbol{u}_n\big) T_k(\rho_n)\phi \,\mathrm{d}\boldsymbol{x} \\
&= -\int_\Omega \big(a\rho_n^\gamma - (2\mu + \lambda)\operatorname{div} \boldsymbol{u}_n\big) \boldsymbol{w}_n \cdot \boldsymbol{\nabla}\phi \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \operatorname{curl} \boldsymbol{u}_n \cdot \operatorname{curl}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} \\
&\quad - \int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}_n \cdot (\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} \\
&= -\int_\Omega \big(a\rho_n^\gamma - (2\mu + \lambda)\operatorname{div} \boldsymbol{u}_n\big) \boldsymbol{w}_n \cdot \boldsymbol{\nabla}\phi \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \operatorname{curl} \boldsymbol{u}_n \cdot L(\phi) \boldsymbol{w}_n \,\mathrm{d}\boldsymbol{x} \\
&\quad - \int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}_n \cdot (\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

Thanks to the previous estimates and convergences (see Prop. 2.6), we are allowed to pass to the limit as $n \to +\infty$

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \big(a\rho_n^\gamma - (2\mu + \lambda)\operatorname{div} \boldsymbol{u}_n\big) T_k(\rho_n)\phi \,\mathrm{d}\boldsymbol{x} \\
&= -\int_\Omega \big(a\overline{\rho^\gamma} - (2\mu + \lambda)\operatorname{div} \boldsymbol{u}\big) \boldsymbol{w} \cdot \boldsymbol{\nabla}\phi \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \operatorname{curl} \boldsymbol{u} \cdot L(\phi) \boldsymbol{w} \,\mathrm{d}\boldsymbol{x} \\
&\quad - \lim_{n\to\infty} \int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot (\phi \boldsymbol{w}) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

An analogous equation can be obtained from the limit momentum equation (2.15) with the test function $\phi \boldsymbol{w}$. It reads:

$$\begin{aligned}
&\int_\Omega \big(a\overline{\rho^\gamma} - (2\mu + \lambda)\operatorname{div} \boldsymbol{u}\big)\ \overline{T_k(\rho)}\phi \,\mathrm{d}\boldsymbol{x} \\
&= -\int_\Omega \big(a\overline{\rho^\gamma} - (2\mu + \lambda)\operatorname{div} \boldsymbol{u}\big) \boldsymbol{w} \cdot \nabla\phi \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \operatorname{curl}(\boldsymbol{u}) \cdot L(\phi) \boldsymbol{w} \,\mathrm{d}\boldsymbol{x} \\
&\quad - \int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla}(\phi \boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot (\phi \boldsymbol{w}) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Comparing the two expressions, we get

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega \big(a\rho_n^\gamma - (2\mu+\lambda)\mathrm{div}\, \boldsymbol{u}_n\big) T_k(\rho_n)\phi \,\mathrm{d}\boldsymbol{x} \\
= \int_\Omega \big(a\overline{\rho^\gamma} - (2\mu+\lambda)\mathrm{div}\, \boldsymbol{u}\big)\overline{T_k(\rho)}\phi \,\mathrm{d}\boldsymbol{x} \\
- \lim_{n\to\infty} \int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla}(\phi \boldsymbol{w}) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Hence it remains to show the two last integrals are equal, which is not direct since we have only weak convergence on $(\rho_n \boldsymbol{u}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{\nabla} \boldsymbol{w}_n)_{n\in\mathbb{N}}$.

This convective term is usually treated with compensated compactness tools by means of Div-Curl and commutator lemmas (see [35] Section 4.4). In the case $\gamma > 3$, a simpler proof is presented in [22] which enables to bypass the use of these tools. Let us first explain the method used in [22] for the case $\gamma > 3$.

We begin with the observation that we can rewrite the integral of the convective term thanks to the continuity equation as

$$\int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} = -\int_\Omega (\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_n \cdot (\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x}.$$

For $\gamma > 3$, $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^q(\Omega)$ for some $q > 6$ and the quantity $((\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_n)_{n\in\mathbb{N}}$ is therefore bounded in $\mathbf{L}^r(\Omega)$ for some $r > \frac{6}{5}$. Let us denote $\boldsymbol{Q} \in \mathbf{L}^r(\Omega)$ its weak limit. Since $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ converges strongly to $\boldsymbol{w}$ in $\mathbf{L}^6(\Omega)$, we obtain (after extracting a subsequence)

$$\lim_{n\to+\infty} \int_\Omega (\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_n \cdot (\phi \boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{Q} \cdot (\phi \boldsymbol{w}) \,\mathrm{d}\boldsymbol{x}.$$

On the other hand, for any fixed test function $\boldsymbol{v} \in \mathbf{W}_0^{1,6}(\Omega)$, it holds

$$\int_\Omega (\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_n \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = -\int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \longrightarrow -\int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \quad \text{as } n \to +\infty$$

combining the weak convergence of $(\rho_n \boldsymbol{u}_n)_{n\in\mathbb{N}}$ in $\mathbf{L}^{\frac{3}{2}}(\Omega)$ and the strong convergence of $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ in $\mathbf{L}^q(\Omega)$, for all $q < 6$. Since the continuity equation is satisfied by the limit pair $(\rho, \boldsymbol{u})$ we have

$$-\int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega (\rho \boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}$$

with $(\rho \boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} \in \mathbf{L}^r(\Omega)$ and thus we identify $\boldsymbol{Q} = (\rho \boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}$, which concludes the proof in the case $\gamma > 3$.

In our case, $\gamma \in (\frac{3}{2}, 3]$, we do not ensure that $\big((\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_n\big)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^r(\Omega)$ for some $r > \frac{6}{5}$. We adapt the previous arguments to our case $\gamma \in (\frac{3}{2}, 3]$ by using a regularization of the velocity $\boldsymbol{u}_n$. Let us first extend $\boldsymbol{u}_n$ and $\boldsymbol{u}$ by 0 on $\mathbb{R}^3 \setminus \Omega$ and introduce the regularized velocities $\boldsymbol{u}_{n,\delta} = \boldsymbol{u}_n * \boldsymbol{\omega}_\delta$ and $\boldsymbol{u}_\delta = \boldsymbol{u} * \boldsymbol{\omega}_\delta$, where $(\boldsymbol{\omega}_\delta)_{\delta>0}$ is a mollifying sequence. By standard properties of the convolution and our *a priori* control of the velocity $\boldsymbol{u}_n$, the following convergences hold (see for instance [15] Lemma 5 p.75 where a regularization of the velocity is also used)

$$\boldsymbol{u}_{n,\delta} \underset{n\to+\infty}{\longrightarrow} \boldsymbol{u}_\delta \quad \text{strongly in } \mathbf{L}^q_{\text{loc}}(\mathbb{R}^3)\ \forall q \in [1,6) \text{ uniformly in } \delta, \tag{2.23}$$

$$\boldsymbol{u}_{n,\delta} \underset{\delta\to 0}{\longrightarrow} \boldsymbol{u}_n \quad \text{strongly in } \mathbf{L}^q_{\text{loc}}(\mathbb{R}^3)\ \forall q \in [1,6) \text{ (uniformly in } n), \tag{2.24}$$

$$\boldsymbol{u}_\delta \underset{\delta\to 0}{\longrightarrow} \boldsymbol{u} \quad \text{strongly in } \mathbf{L}^6_{\text{loc}}(\mathbb{R}^3). \tag{2.25}$$

Since $\operatorname{div}(\rho_n \boldsymbol{u}_n) = 0$, we then have, for any $\delta > 0$:

$$\begin{aligned}\int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^3} \boldsymbol{u}_{n,\delta} \otimes (\rho_n \boldsymbol{u}_n) : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + R_1^{n,\delta}\\ &= -\int_{\mathbb{R}^3} \mathbf{div}(\boldsymbol{u}_{n,\delta} \otimes \rho_n \boldsymbol{u}_n) \cdot (\phi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + R_1^{n,\delta}\\ &= -\int_{\mathbb{R}^3} (\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_{n,\delta} \cdot (\phi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + R_1^{n,\delta}\end{aligned}$$

where

$$R_1^{n,\delta} = \int_{\mathbb{R}^3} (\boldsymbol{u}_n - \boldsymbol{u}_{n,\delta}) \otimes (\rho_n \boldsymbol{u}_n) : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x}.$$

Since $(\rho_n \boldsymbol{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^p(\Omega)$ for some $p > \frac{6}{5}$, $(\boldsymbol{\nabla} \boldsymbol{w}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^s(\Omega)^3$ for any $s \in (1,+\infty)$, then the following inequality holds, for some triple (p,q,s), such that $p > \frac{6}{5}$, $s > 1$, $q < 6$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$:

$$\begin{aligned}|R_1^{n,\delta}| &\le C \|\rho_n \boldsymbol{u}_n\|_{\mathbf{L}^p(\Omega)} \|\boldsymbol{\nabla}(\phi \boldsymbol{w}_n)\|_{\mathbf{L}^s(\Omega)^3} \|\boldsymbol{u}_n - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\mathbb{R}^3)}\\ &\le C \|\boldsymbol{u}_n - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\mathbb{R}^3)}.\end{aligned}$$

As a consequence:

$$\limsup_{n\to+\infty} |R_1^{n,\delta}| \le C \limsup_{n\to+\infty} \|\boldsymbol{u}_n - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\mathbb{R}^3)}. \tag{2.26}$$

Thanks to the regularization of the velocity, we ensure that $(\boldsymbol{\nabla} \boldsymbol{u}_{n,\delta})_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^6_{\text{loc}}(\mathbb{R}^3)^3$. The sequence $\big((\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_{n,\delta}\big)_{n\in\mathbb{N}} = (\boldsymbol{Q}_{n,\delta})_{n\in\mathbb{N}}$ is then bounded in $\mathbf{L}^r(\Omega)$, for some $r > 1$ and up to the extraction of a subsequence, it weakly converges in $\mathbf{L}^r(\Omega)$ towards some function $\boldsymbol{Q}_\delta \in \mathbf{L}^r(\Omega)$. We have

$$\begin{aligned}\int_\Omega \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} &= -\int_\Omega (\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_{n,\delta} \cdot (\phi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + R_1^{n,\delta}\\ &= -\int_\Omega \boldsymbol{Q}_\delta \cdot (\phi \boldsymbol{w})\,\mathrm{d}\boldsymbol{x} + R_1^{n,\delta} + R_2^{n,\delta}\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}R_2^{n,\delta} &= -\int_\Omega \boldsymbol{Q}_{n,\delta} \cdot (\phi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{Q}_\delta \cdot (\phi \boldsymbol{w})\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega \big(\boldsymbol{Q}_\delta - \boldsymbol{Q}_{n,\delta}\big) \cdot (\phi \boldsymbol{w})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \phi\, \boldsymbol{Q}_{n,\delta} \cdot \big(\boldsymbol{w} - \boldsymbol{w}_n\big)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Since $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ converges strongly to $\boldsymbol{w}$ in $\mathbf{L}^q(\Omega)$, for all $q \in (1,+\infty)$, we have

$$|R_2^{n,\delta}| \to 0 \quad \text{as } n \to +\infty \text{ for any fixed } \delta > 0. \tag{2.28}$$

We now want to show that $\boldsymbol{Q}_\delta = (\rho\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_\delta$. To that end, let us consider a fixed test function $\boldsymbol{v} \in \mathbf{W}_0^{1,\infty}(\Omega)$ and write (again thanks to the fact that $\operatorname{div}(\rho_n \boldsymbol{u}_n) = 0$)

$$
\begin{aligned}
\int_\Omega (\rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla})\boldsymbol{u}_{n,\delta}\cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} &= -\int_\Omega \boldsymbol{u}_{n,\delta}\otimes(\rho_n\boldsymbol{u}_n) : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\
&= -\int_\Omega \boldsymbol{u}_\delta \otimes(\rho_n\boldsymbol{u}_n) : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \tilde{R}_2^{n,\delta} \\
&= \int_\Omega \mathbf{div}(\boldsymbol{u}_\delta\otimes \rho_n\boldsymbol{u}_n)\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \tilde{R}_2^{n,\delta} \\
&= \int_\Omega (\rho\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_\delta\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \tilde{R}_2^{n,\delta}
\end{aligned}
$$

with

$$
\tilde{R}_2^{n,\delta} = \int_\Omega (\boldsymbol{u}_\delta - \boldsymbol{u}_{n,\delta})\otimes(\rho_n\boldsymbol{u}_n) : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}
$$

which tends to 0 (uniformly with respect to δ) as $n\to+\infty$, since $(\rho_n\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges weakly to $\rho\boldsymbol{u}$ in $\mathbf{L}^{q_1}(\Omega)$ for some $q_1 > \frac{6}{5}$, and $(\boldsymbol{u}_{n,\delta})_{n\in\mathbb{N}}$ converges strongly to $\boldsymbol{u}_\delta$ (uniformly with respect to δ) in $\mathbf{L}^{q_2}(\Omega)$ for any $q_2 < 6$. As a consequence, we identify $\boldsymbol{Q}_\delta = (\rho\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_\delta$. Now, back to (2.27), since at the limit $\operatorname{div}(\rho\boldsymbol{u}) = 0$, we have:

$$
\begin{aligned}
\int_\Omega \rho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n : \boldsymbol{\nabla}(\phi\boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} &= -\int_\Omega (\rho\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_\delta\cdot(\phi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x} + R_1^{n,\delta} + R_2^{n,\delta} \\
&= \int_\Omega \rho\boldsymbol{u}_\delta\otimes\boldsymbol{u} : \boldsymbol{\nabla}(\phi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x} + R_1^{n,\delta} + R_2^{n,\delta} \\
&= \int_\Omega \rho\boldsymbol{u}\otimes\boldsymbol{u} : \boldsymbol{\nabla}(\phi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x} + R_1^{n,\delta} + R_2^{n,\delta} + R_3^\delta
\end{aligned}
$$

where

$$
R_3^\delta = \int_\Omega (\boldsymbol{u}_\delta - \boldsymbol{u})\otimes(\rho\boldsymbol{u}) : \boldsymbol{\nabla}(\phi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}.
$$

Combining (2.26) and (2.28), we get that for any fixed $\delta > 0$:

$$
\limsup_{n\to+\infty}\left|\int_\Omega \rho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n : \boldsymbol{\nabla}(\phi\boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho\boldsymbol{u}\otimes\boldsymbol{u} : \boldsymbol{\nabla}(\phi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}\right| \le C\limsup_{n\to+\infty}\|\boldsymbol{u}_n - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\mathbb{R}^3)} + |R_3^\delta|,
$$

for some $q < 6$. By (2.25), we have $R_3^\delta \to 0$ as $\delta\to 0$. Hence, by the uniform in n convergence of $(\boldsymbol{u}_{n,\delta})_{\delta>0}$ towards $\boldsymbol{u}_n$ as $\delta \to 0$ (2.24), letting δ tend to 0 yields:

$$
\lim_{n\to+\infty}\int_\Omega \rho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n : \boldsymbol{\nabla}(\phi\boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\boldsymbol{u}\otimes\boldsymbol{u} : \boldsymbol{\nabla}(\phi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}.
$$

We finally conclude that

$$
\lim_{n\to+\infty}\int_\Omega \big(a\rho_n^\gamma - (2\mu+\lambda)\operatorname{div}\boldsymbol{u}_n\big)T_k(\rho_n)\phi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(a\overline{\rho^\gamma} - (2\mu+\lambda)\operatorname{div}\boldsymbol{u}\big)\overline{T_k(\rho)}\phi\,\mathrm{d}\boldsymbol{x}.
$$

□

2.4.2 Strong convergence of the density

Let us begin this subsection with a brief sketch of the general strategy that we will employ. Concentration phenomena being excluded, the only mechanism which can prevent the strong convergence is the presence of oscillations. We need to prove that we control these oscillations. For large values of γ, namely $\gamma \geq 2$, one can show an "improved" version of the weak compactness of the effective viscous flux (2.21) where $T_k(\rho_n)$ (resp. $\overline{T_k(\rho)}$) is replaced by ρ_n (resp ρ). Then, passing to the limit $n \to +\infty$ in the renormalized continuity equation

$$\mathrm{div}((\rho_n \ln \rho_n)\, \boldsymbol{u}_n) = -\rho_n \mathrm{div}\, \boldsymbol{u}_n \quad \text{in } \mathcal{D}'(\mathbb{R}^3)$$

we get

$$\mathrm{div}(\overline{\rho \ln \rho}\ \boldsymbol{u}) = -\overline{\rho \mathrm{div}\, \boldsymbol{u}} \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

On the other hand, applying the renormalization theory of Di Perna-Lions (Lemma 2.1) on the limit $\rho \in \mathrm{L}^{3(\gamma-1)}(\Omega)$, $\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)$ we also have

$$\mathrm{div}((\rho \ln \rho)\, \boldsymbol{u}) = -\rho \mathrm{div}\, \boldsymbol{u}. \tag{2.29}$$

Subtracting this equation from the previous one, we arrive at

$$\mathrm{div}\big((\overline{\rho \ln \rho} - \rho \ln \rho)\ \boldsymbol{u}\big) = \rho \mathrm{div}\, \boldsymbol{u} - \overline{\rho \mathrm{div}\, \boldsymbol{u}} \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

The weak compactness property of the effective viscous flux yields

$$\mathrm{div}\big((\overline{\rho \ln \rho} - \rho \ln \rho)\ \boldsymbol{u}\big) = \frac{a}{2\mu + \lambda}\Big(\rho \overline{\rho^\gamma} - \overline{\rho^{\gamma+1}}\Big) \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

Integrating in space, we end up with the identity $\rho \overline{\rho^\gamma} = \overline{\rho^{\gamma+1}}$ a.e., from which the strong convergence of the density follows by invoking the monotonicity of the pressure (Minty's trick).

In the arguments presented above, one of the key point is to write (2.29) which requires from the theory of Di Perna and Lions that $\rho \in \mathrm{L}^2(\Omega)$ (see Remark 2.4). The previous proof can be adapted in the case $\gamma \in (\frac{5}{3}, 2)$ using a "weaker" version of the effective viscous flux identity where ρ_n is essentially replaced by ρ_n^α for some $\alpha \in (0,1)$. This is the case initially demonstrated by Lions in [33]. For smaller values of γ, i.e. $\frac{3}{2} < \gamma \leq \frac{5}{3}$, we do not ensure *a priori* (2.29) since ρ does not belong to $\mathrm{L}^2(\Omega)$. The idea of Feireisl [14] (adapted then by Novo and Novotný in the stationary case) is to work on the truncated variable $T_k(\rho)$, defined in (2.20) which is bounded for fixed k (and thus in $\mathrm{L}^2(\Omega)$). With similar arguments as before, one may then show the strong convergence of $(T_k(\rho_n))_{n\in\mathbb{N}}$ to $T_k(\rho)$ (uniformly with respect to n in $\mathrm{L}^{\gamma+1}(\Omega)$). Combining finally this result with the strong convergence of the truncated variables as $k \to +\infty$ (see Lemma 2.10 below), we will get the strong convergence of $(\rho_n)_{n\in\mathbb{N}}$.

Properties of the truncation operators T_k.

Lemma 2.10. *Under the assumptions of Proposition 2.9, there exists a constant C such that the following inequality holds for all $1 \leq q < 3(\gamma-1)$, $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$:*

$$\|\overline{T_k(\rho)} - \rho\|_{\mathrm{L}^q(\Omega)} + \|T_k(\rho) - \rho\|_{\mathrm{L}^q(\Omega)} + \|T_k(\rho_n) - \rho_n\|_{\mathrm{L}^q(\Omega)} \leq C k^{\frac{1}{3(\gamma-1)} - \frac{1}{q}}. \tag{2.30}$$

Consequently, as $k \to +\infty$, the sequences $(\overline{T_k(\rho)})_{k\in\mathbb{N}^*}$ and $(T_k(\rho))_{k\in\mathbb{N}^*}$ both converge strongly to ρ in $\mathrm{L}^q(\Omega)$ for all $q \in [1, 3(\gamma-1))$.

Proof. As a consequence of the inequality

$$|\{\rho_n \geq k\}| \leq \frac{1}{k} \int_\Omega \rho_n \,\mathrm{d}\boldsymbol{x} = |\Omega| \frac{\rho_n^*}{k} \leq \frac{C}{k}$$

we deduce by Hölder's inequality that for any $q < 3(\gamma - 1)$

$$\begin{aligned}
\|T_k(\rho_n) - \rho_n\|_{\mathrm{L}^q(\Omega)} &= \|(T_k(\rho_n) - \rho_n)\mathbf{1}_{\{\rho_n \geq k\}}\|_{\mathrm{L}^q(\Omega)} \\
&\leq \|\rho_n \mathbf{1}_{\{\rho_n \geq k\}}\|_{\mathrm{L}^q(\Omega)} \\
&\leq C k^{\frac{1}{3(\gamma-1)} - \frac{1}{q}} \|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)} \\
&\leq C k^{\frac{1}{3(\gamma-1)} - \frac{1}{q}},
\end{aligned}$$

where the constant C only depends on q and the uniform bounds on the sequences $(\rho_n^\star)_{n\in\mathbb{N}}$ and $(\|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}})_{n\in\mathbb{N}}$. Doing the same with the limit density ρ, we get

$$\|T_k(\rho) - \rho\|_{\mathrm{L}^q(\Omega)} \leq C k^{\frac{1}{3(\gamma-1)} - \frac{1}{q}}.$$

Finally, we have:

$$\begin{aligned}
\|\overline{T_k(\rho)} - \rho\|_{\mathrm{L}^q(\Omega)} &\leq \liminf_{n\to+\infty} \|T_k(\rho_n) - \rho_n\|_{\mathrm{L}^q(\Omega)} \\
&\leq \limsup_{n\to+\infty} \|T_k(\rho_n) - \rho_n\|_{\mathrm{L}^q(\Omega)} \\
&\leq C k^{\frac{1}{3(\gamma-1)} - \frac{1}{q}}.
\end{aligned}$$

which ends the proof. □

Lemma 2.11. *Under the assumptions of Proposition 2.9, there exists a constant C such that the following estimate holds:*

$$\sup_{k>1} \limsup_{n\to+\infty} \|T_k(\rho_n) - T_k(\rho)\|_{\mathrm{L}^{\gamma+1}(\Omega)} \leq C. \tag{2.31}$$

Proof. First of all, observe that for all $r_1, r_2 \geq 0$,

$$|T_k(r_1) - T_k(r_2)|^{\gamma+1} \leq (r_1^\gamma - r_2^\gamma)(T_k(r_1) - T_k(r_2))$$

and thus

$$\begin{aligned}
\limsup_{n\to+\infty} \int_\Omega |T_k(\rho_n) - T_k(\rho)|^{\gamma+1} &\leq \limsup_{n\to+\infty} \int_\Omega (\rho_n^\gamma - \rho^\gamma)(T_k(\rho_n) - T_k(\rho)) \\
&\leq \int_\Omega \big(\overline{\rho^\gamma T_k(\rho)} - \overline{\rho^\gamma}\ \overline{T_k(\rho)}\big) + \int_\Omega \big(\overline{\rho^\gamma} - \rho^\gamma\big)\big(\overline{T_k(\rho)} - T_k(\rho)\big).
\end{aligned}$$

Invoking the convexity of the functions $t \mapsto t^\gamma$ and $t \mapsto -T_k(t)$, we have $\overline{\rho^\gamma} \geq \rho^\gamma$ and $\overline{T_k(\rho)} \leq T_k(\rho)$ so that

$$\limsup_{n\to+\infty} \int_\Omega |T_k(\rho_n) - T_k(\rho)|^{\gamma+1} \leq \int_\Omega \big(\overline{\rho^\gamma T_k(\rho)} - \overline{\rho^\gamma}\ \overline{T_k(\rho)}\big).$$

We can now use the weak compactness property satisfied by the effective viscous flux (2.21):

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega |T_k(\rho_n)-T_k(\rho)|^{\gamma+1}\\
&\le \frac{2\mu+\lambda}{a}\limsup_{n\to+\infty}\int_\Omega \big(T_k(\rho_n)-\overline{T_k(\rho)}\big)\operatorname{div}\boldsymbol{u}_n \\
&\le \frac{2\mu+\lambda}{a}\limsup_{n\to+\infty}\int_\Omega \big(T_k(\rho_n)-T_k(\rho)\big)\operatorname{div}\boldsymbol{u}_n+\frac{2\mu+\lambda}{a}\limsup_{n\to+\infty}\int_\Omega \big(T_k(\rho)-\overline{T_k(\rho)}\big)\operatorname{div}\boldsymbol{u}_n\\
&\le C\limsup_{n\to+\infty}\|T_k(\rho_n)-T_k(\rho)\|_{\mathrm{L}^2(\Omega)}.
\end{aligned} \tag{2.32}$$

Where C depends on the uniform bound on the sequence $(\|\operatorname{div}\boldsymbol{u}_n\|_{\mathrm{L}^2(\Omega)})_{n\in\mathbb{N}}$. Since $\gamma+1>2$, we obtain thanks to Hölder and Young inequalities

$$\limsup_{n\to+\infty}\int_\Omega |T_k(\rho_n)-T_k(\rho)|^{\gamma+1}\le C+\frac{1}{2}\limsup_{n\to+\infty}\|T_k(\rho_n)-T_k(\rho)\|^{\gamma+1}_{\mathrm{L}^{\gamma+1}(\Omega)}$$

which achieves the proof. □

Renormalization equation associated with T_k. For any $k\in\mathbb{N}^*$, define

$$L_k(t)=\begin{cases} t(\ln t-\ln k-1), & \text{if } t\in[0,k),\\ -k, & \text{if } t\in[k,+\infty),\end{cases} \tag{2.33}$$

which belongs to $\mathcal{C}^0([0,+\infty))\cap\mathcal{C}^1((0,+\infty))$ and which is such that

$$tL_k'(t)-L_k(t)=T_k(t)\quad \forall t\in[0,+\infty).$$

Remark 2.6. *Note that function L_k can be seen as a truncated version of the function $L(t)=t\ln t$ used in (2.29) for large γ, up to the addition of the linear function $t\mapsto-(\ln k+1)t$.*

Proposition 2.12. *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$. Assume that $\gamma\in(\frac{3}{2},3]$. Let $(\rho_n,\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be the sequence defined in Theorem 2.2 and let $(\rho,\boldsymbol{u})\in\mathrm{L}^{3(\gamma-1)}(\Omega)\times\mathbf{H}^1_0(\Omega)$ be its limit defined in Proposition 2.6. Then, for all $k\in\mathbb{N}^*$, the following equations hold:*

$$\operatorname{div}(L_k(\rho_n)\boldsymbol{u}_n)+T_k(\rho_n)\operatorname{div}\boldsymbol{u}_n=0,\qquad \textit{in } \mathcal{D}'(\mathbb{R}^3),\quad \forall n\in\mathbb{N}. \tag{2.34}$$

$$\operatorname{div}(L_k(\rho)\boldsymbol{u})+T_k(\rho)\operatorname{div}\boldsymbol{u}=0\qquad \textit{in } \mathcal{D}'(\mathbb{R}^3). \tag{2.35}$$

Proof. Since $L_k\in\mathcal{C}^0([0,+\infty))\cap\mathcal{C}^1((0,+\infty))$ satisfies (2.5), and since $(\rho_n,\boldsymbol{u}_n)$ is a renormalized solution of (1.1)-(1.2)-(1.3), in the sense of Definition 2.1, equation (2.34) holds true.

Let us prove that a similar equation is also satisfied for the limit couple $(\rho,\boldsymbol{u})$. Using the renormalization property (2.7) satisfied by $(\rho_n,\boldsymbol{u}_n)$ for the truncated function T_M, $M\in\mathbb{N}^*$, we obtain:

$$\operatorname{div}\big(T_M(\rho_n)\boldsymbol{u}_n\big)=-\big[\rho_n[T_M]'_+(\rho_n)-T_M(\rho_n)\big]\operatorname{div}\boldsymbol{u}_n\quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

which yields as $n \to +\infty$

$$\operatorname{div}\big(\overline{T_M(\rho)}\boldsymbol{u}\big) = -\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]\operatorname{div}\boldsymbol{u}} \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{2.36}$$

For $k \in \mathbb{N}^*$ and $\delta > 0$, we introduce the regularized function $L_{k,\delta}$ defined as

$$L_{k,\delta}(t) = L_k(t+\delta), \tag{2.37}$$

the derivative of which is bounded close to 0 unlike L_k. Applying Lemma 2.1 (and the second part of Remark 2.4) to the pair $(\overline{T_M(\rho)}, \boldsymbol{u})$ (justified since $\overline{T_M(\rho)} \in L^\infty(\Omega)$ for M fixed) with the function $L_{k,\delta}$ and the source term $g = -\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]\operatorname{div}\boldsymbol{u}} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3)$, we get:

$$\begin{aligned}\operatorname{div}\big(L_{k,\delta}\big(\overline{T_M(\rho)}\big)\boldsymbol{u}\big) &+ T_{k,\delta}\big(\overline{T_M(\rho)}\big)\operatorname{div}\boldsymbol{u} \\ &= -L'_{k,\delta}\big(\overline{T_M(\rho)}\big)\,\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]\operatorname{div}\boldsymbol{u}} \quad \text{in } \mathcal{D}'(\mathbb{R}^3)\end{aligned} \tag{2.38}$$

where

$$T_{k,\delta}(t) = tL'_{k,\delta}(t) - L_{k,\delta}(t).$$

We now have to pass to the limits $M \to +\infty$, $\delta \to 0^+$. Lemma 2.10 yields the strong convergence of $\overline{T_M(\rho)}$ to ρ as $M \to +\infty$. As a consequence, the left-hand side of (2.38) converges in $\mathcal{D}'(\mathbb{R}^3)$ to

$$\operatorname{div}\big(L_{k,\delta}(\rho)\boldsymbol{u}\big) + T_{k,\delta}(\rho)\operatorname{div}\boldsymbol{u}.$$

Regarding the right-hand side

$$-L'_{k,\delta}\big(\overline{T_M(\rho)}\big)\,\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]\operatorname{div}\boldsymbol{u}},$$

since $L'_{k,\delta}(t) = 0$ for $t > k$, we estimate its L^1 norm as follows

$$\begin{aligned}&\left|\int_\Omega L'_{k,\delta}\big(\overline{T_M(\rho)}\big)\,\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]\operatorname{div}\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}\right| \\ &\qquad\qquad \leq \max_{t\in[0,k]}|L'_{k,\delta}(t)| \int_\Omega \left|\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]\operatorname{div}\boldsymbol{u}}\right| \mathbf{1}_{\Omega_{M,k}}\,\mathrm{d}\boldsymbol{x}\end{aligned}$$

where $\Omega_{M,k} = \mathbf{1}_{\{\overline{T_M(\rho)} \leq k\}}$. We then have

$$\begin{aligned}\max_{t\in[0,k]}|L'_{k,\delta}(t)| &\int_\Omega \left|\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]\operatorname{div}\boldsymbol{u}}\right| \mathbf{1}_{\Omega_{M,k}}\,\mathrm{d}\boldsymbol{x} \\ &\leq \max_{t\in[0,k]}|L'_{k,\delta}(t)| \liminf_{n\to+\infty}\int_\Omega \big|\big[\rho_n[T_M]'_+(\rho_n) - T_M(\rho_n)\big]\operatorname{div}\boldsymbol{u}_n\big|\,\mathbf{1}_{\Omega_{M,k}}\,\mathrm{d}\boldsymbol{x} \\ &\leq \max_{t\in[0,k]}|L'_{k,\delta}(t)| \limsup_{n\to+\infty}\int_\Omega \big|\big[\rho_n[T_M]'_+(\rho_n) - T_M(\rho_n)\big]\operatorname{div}\boldsymbol{u}_n\big|\,\mathbf{1}_{\Omega_{M,k}}\,\mathrm{d}\boldsymbol{x} \\ &\leq C\max_{t\in[0,k]}|L'_{k,\delta}(t)| \limsup_{n\to+\infty}\|T_M(\rho_n)\mathbf{1}_{\Omega_{M,k}\cap\{\rho_n\geq M\}}\|_{\mathrm{L}^2(\Omega)},\end{aligned} \tag{2.39}$$

since the sequence $(\|\operatorname{div}\boldsymbol{u}_n\|_{\mathrm{L}^2(\Omega)})_{n\in\mathbb{N}}$ is bounded. We already know that $T_M(\rho_n)\mathbf{1}_{\Omega_{M,k}\cap\{\rho_n\geq M\}}$ is controlled in $\mathrm{L}^1(\Omega)$ since

$$\begin{aligned}\|T_M(\rho_n)\mathbf{1}_{\Omega_{M,k}\cap\{\rho_n\geq M\}}\|_{\mathrm{L}^1(\Omega)} &\leq \|\rho_n\mathbf{1}_{\{\rho_n\geq M\}}\|_{\mathrm{L}^1(\Omega)} \\ &\leq CM^{\frac{1}{3(\gamma-1)}-1}\|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)} \\ &\leq CM^{\frac{1}{3(\gamma-1)}-1},\end{aligned}$$

where the constant C only depends on the uniform bounds on $(\rho_n^\star)_{n\in\mathbb{N}}$ and $(\|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}})_{n\in\mathbb{N}}$. Therefore, by an interpolation inequality, we obtain:

$$\begin{aligned}
&\limsup_{n\to+\infty} \|T_M(\rho_n)\mathcal{X}_{\Omega_{M,k}\cap\{\rho_n\geq M\}}\|_{\mathrm{L}^2(\Omega)} \\
&\leq C \limsup_{n\to+\infty} \|T_M(\rho_n)\mathcal{X}_{\Omega_{M,k}\cap\{\rho_n\geq M\}}\|_{\mathrm{L}^1(\Omega)}^{\frac{\gamma-1}{2\gamma}} \|T_M(\rho_n)\mathcal{X}_{\Omega_{M,k}\cap\{\rho_n\geq M\}}\|_{\mathrm{L}^{\gamma+1}(\Omega)}^{\frac{\gamma+1}{2\gamma}} \\
&\leq C M^{\frac{\gamma-1}{2\gamma}\left(\frac{1}{3(\gamma-1)}-1\right)} \limsup_{n\to+\infty} \Big(\|\big(T_M(\rho_n)-\overline{T_M(\rho)}\big)\mathcal{X}_{\Omega_{M,k}}\|_{\mathrm{L}^{\gamma+1}(\Omega)} + \|\overline{T_M(\rho)}\mathcal{X}_{\Omega_{M,k}}\|_{\mathrm{L}^{\gamma+1}(\Omega)} \Big)^{\frac{\gamma+1}{2\gamma}} \\
&\leq C M^{\frac{\gamma-1}{2\gamma}\left(\frac{1}{3(\gamma-1)}-1\right)} \limsup_{n\to+\infty} \Big(\|\big(T_M(\rho_n)-\overline{T_M(\rho)}\big)\|_{\mathrm{L}^{\gamma+1}(\Omega)} + k|\Omega|^{\frac{1}{\gamma+1}} \Big)^{\frac{\gamma+1}{2\gamma}}
\end{aligned}$$

Thanks to Lemma 2.11, we deduce that

$$\limsup_{n\to+\infty} \|T_M(\rho_n)\mathcal{X}_{\Omega_{M,k}\cap\{\rho_n\geq M\}}\|_{\mathrm{L}^2(\Omega)} \leq C(k,\Omega) M^{\frac{\gamma-1}{2\gamma}\left(\frac{1}{3(\gamma-1)}-1\right)} \to 0 \quad \text{as } M\to+\infty.$$

Injecting in (2.39), we get

$$\max_{t\in[0,k]} |L'_{k,\delta}(t)| \int_\Omega \left| \overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]\operatorname{div}\boldsymbol{u}} \right| \mathcal{X}_{\Omega_{M,k}} \,\mathrm{d}\boldsymbol{x} \ \to 0 \quad \text{as } M\to+\infty.$$

Note that to get this result, we have been forced to regularize the function L_k (see (2.37)) in order to control its derivative close to 0. Hence, passing to the limit $M\to+\infty$ in (2.38) we get

$$\operatorname{div}\big(L_{k,\delta}(\rho)\boldsymbol{u}\big) + T_{k,\delta}(\rho)\operatorname{div}\boldsymbol{u} = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^3), \quad \forall\, k\in\mathbb{N}^*, \delta>0.$$

Now, observe that for all $t\in[0,+\infty)$

$$\begin{aligned}
L_{k,\delta}(t) &\underset{\delta\to0^+}{\longrightarrow} L_k(t), \\
T_{k,\delta}(t) = tL'_{k,\delta}(t) - L_{k,\delta}(t) &\underset{\delta\to0^+}{\longrightarrow} tL'_k(t) - L_k(t) = T_k(t).
\end{aligned}$$

Moreover, since for all $t\in[0,+\infty)$ and $\delta\in(0,1)$, we have $|L_{k,\delta}(t)|\leq k$ and

$$\begin{aligned}
|T_{k,\delta}(t)| &= |T_k(t+\delta) - \delta L'_k(t+\delta)| \\
&\leq k + \delta|\ln(t+\delta) - \ln\, k|\mathcal{X}_{\{t+\delta\leq k\}} \leq k + \delta\Big(|\ln\,\delta| + \frac{t}{\delta} + \ln\, k\Big)\mathcal{X}_{\{t+\delta\leq k\}} \leq C(k),
\end{aligned}$$

we can pass to the limit $\delta\to0^+$ thanks to the Lebesgue Dominated Convergence Theorem to get

$$\operatorname{div}\big(L_k(\rho)\boldsymbol{u}\big) + T_k(\rho)\operatorname{div}\boldsymbol{u} = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^3),$$

for all $k\in\mathbb{N}^*$ which concludes the proof. □

Strong convergence of the density

Proposition 2.13. *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$. Assume that $\gamma \in (\frac{3}{2}, 3]$. Let $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ be the sequence defined in Theorem 2.2 and let $(\rho, \boldsymbol{u}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}^1_0(\Omega)$ be its limit defined in Proposition 2.6. Up to extraction, the sequence $(\rho_n)_{n\in\mathbb{N}}$ strongly converges towards ρ in $\mathrm{L}^q(\Omega)$ for all $q \in [1, 3(\gamma-1))$.*

Proof. Integrating the renormalized continuity equations (2.34) and (2.35) and summing, one obtains:

$$\int_\Omega T_k(\rho_n) \mathrm{div}\, \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} - \int_\Omega T_k(\rho) \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} = 0, \qquad \forall n \in \mathbb{N}.$$

We then use this identity in inequality (2.32) to deduce that

$$\begin{aligned}
\limsup_{n\to+\infty} \int_\Omega |T_k(\rho_n) - T_k(\rho)|^{\gamma+1} \,\mathrm{d}\boldsymbol{x} &\\
\leq \frac{2\mu+\lambda}{a} \limsup_{n\to+\infty} \int_\Omega \big(T_k(\rho_n) - \overline{T_k(\rho)}\big) \mathrm{div}\, \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} &\\
= \frac{2\mu+\lambda}{a} \int_\Omega (T_k(\rho) - \overline{T_k(\rho)})\, \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} + \limsup_{n\to+\infty} \Big(\int_\Omega T_k(\rho_n) \mathrm{div}\, \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} - \int_\Omega T_k(\rho) \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} \Big) &\\
= \frac{2\mu+\lambda}{a} \int_\Omega (T_k(\rho) - \overline{T_k(\rho)})\, \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}. &
\end{aligned}$$

Using Lemma 2.10 we infer that

$$\begin{aligned}
\left| \int_\Omega (T_k(\rho) - \overline{T_k(\rho)})\, \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} \right| &\leq \|\mathrm{div}\, \boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} \|T_k(\rho) - \overline{T_k(\rho)}\|_{\mathrm{L}^2(\Omega)} \\
&\leq C \|T_k(\rho) - \overline{T_k(\rho)}\|_{\mathrm{L}^1(\Omega)}^{\frac{\gamma-1}{2\gamma}} \|T_k(\rho) - \overline{T_k(\rho)}\|_{\mathrm{L}^{\gamma+1}(\Omega)}^{\frac{\gamma+1}{2\gamma}} \\
&\leq C \Big(\|T_k(\rho) - \rho\|_{\mathrm{L}^1(\Omega)} + \|\overline{T_k(\rho)} - \rho\|_{\mathrm{L}^1(\Omega)} \Big)^{\frac{\gamma-1}{2\gamma}} \|T_k(\rho) - \overline{T_k(\rho)}\|_{\mathrm{L}^{\gamma+1}(\Omega)}^{\frac{\gamma+1}{2\gamma}} \\
&\leq C k^{\frac{\gamma-1}{2\gamma}\left(\frac{1}{3(\gamma-1)} - 1\right)} \|T_k(\rho) - \overline{T_k(\rho)}\|_{\mathrm{L}^{\gamma+1}(\Omega)}^{\frac{\gamma+1}{2\gamma}} \\
&\leq C k^{\frac{\gamma-1}{2\gamma}\left(\frac{1}{3(\gamma-1)} - 1\right)} \Big(\limsup_{n\to+\infty} \|T_k(\rho_n) - T_k(\rho)\|_{\mathrm{L}^{\gamma+1}(\Omega)} \Big)^{\frac{\gamma+1}{2\gamma}}.
\end{aligned}$$

As a consequence of Lemma 2.11, we have

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \int_\Omega |T_k(\rho_n) - T_k(\rho)|^{\gamma+1} \,\mathrm{d}\boldsymbol{x} = 0,$$

and thus

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \|T_k(\rho_n) - T_k(\rho)\|_{\mathrm{L}^1(\Omega)} = 0. \tag{2.40}$$

We conclude to the strong convergence of the density by writing

$$\|\rho - \rho_n\|_{\mathrm{L}^1(\Omega)} \leq \|\rho_n - T_k(\rho_n)\|_{\mathrm{L}^1(\Omega)} + \|T_k(\rho_n) - T_k(\rho)\|_{\mathrm{L}^1(\Omega)} + \|T_k(\rho) - \rho\|_{\mathrm{L}^1(\Omega)}.$$

Passing to the limit superior $n \to +\infty$ in this inequality, using again Lemma 2.10 and the previous estimate (2.40) to then pass to the limit $k \to +\infty$ yields the strong convergence of the density in $\mathrm{L}^1(\Omega)$ and therefore in $\mathrm{L}^q(\Omega)$ for all $q \in [1, 3(\gamma-1))$. ☐

2.5 Elements for the construction of weak solutions

The previous subsections were concerned with the stability of weak solutions of Problem (1.1)-(1.2)-(1.3). An important step is the effective construction of such weak solutions by means of successive approximations. This construction is sketched by Lions in [33] for the case $\gamma > \frac{5}{3}$ and detailed by Novo and Novotný in [34] for the case $\gamma > \frac{3}{2}$. The approximation procedure is usually decomposed into three steps:

- addition in the momentum equation of an *artificial pressure* term $\delta \boldsymbol{\nabla} \rho^{\Gamma}$ with Γ sufficiently large, namely $\Gamma > 3$;
- addition of a *relaxation term* $\alpha(\rho - \rho^{\star})$ in the mass equation in order to ensure that the total mass constraint (1.3) is satisfied at the approximate level;
- addition of a diffusion or *regularization term* (*e.g.* $-\varepsilon \Delta \rho$) in the mass equation which regularizes the density.

As a consequence of the modification of the mass equation, the momentum equation can also involve additional perturbation terms depending on α and ε, in order to ensure the preservation of the energy inequality (see details in [34] or [35]). The parameters $\delta, \alpha, \varepsilon$ being fixed, the existence of weak solutions is obtained by a fixed point argument. Then, the proof consists in passing to the limit successively with respect to ε, α and then finally with respect to δ.

In the next section, we present our numerical scheme which essentially reproduces at the discrete level the previous three approximation terms. Nevertheless, the parameters ε, α, δ are no more independent in the discrete case, they shall all depend on the mesh size and converge to 0 as the mesh size tends to 0. In that sense, the convergence result that we obtain in the next sections can be seen as an alternative proof of existence of weak solutions to the stationary problem (1.1)-(1.2)-(1.3).

3 The discrete setting, presentation of the numerical scheme

3.1 Meshes and discretization spaces

Let Ω be an open bounded connected subset of $\mathbb{R}^d$, $d = 2$ or 3. We assume that Ω is polygonal if $d = 2$ and polyhedral if $d = 3$.

Definition 3.1 (Staggered mesh)**.** *A staggered discretization of Ω, denoted by $\mathcal{D}$, is given by a pair $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, where:*

- *$\mathcal{M}$, the so-called primal mesh, is a finite family composed of non empty simplices. The primal mesh $\mathcal{M}$ is assumed to form a partition of Ω : $\overline{\Omega} = \cup_{K \in \mathcal{M}} \overline{K}$. For any open simplex $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K, which is the union of cell faces. We denote by $\mathcal{E}$ the set of faces of the mesh, and we suppose that two neighboring cells share a whole face: for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K, L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \sigma$; we denote in the latter case $\sigma = K|L$. We denote by $\mathcal{E}_{\rm ext}$ and $\mathcal{E}_{\rm int}$ the set of external and internal faces: $\mathcal{E}_{\rm ext} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$ and $\mathcal{E}_{\rm int} = \mathcal{E} \setminus \mathcal{E}_{\rm ext}$. For $K \in \mathcal{M}$, $\mathcal{E}(K)$ stands for the set of faces of K. The unit vector normal to $\sigma \in \mathcal{E}(K)$ outward K is denoted by $\boldsymbol{n}_{K,\sigma}$. In the following, the notation $|K|$ or $|\sigma|$ stands indifferently for the d-dimensional or the $(d-1)$-dimensional measure of the subset K of $\mathbb{R}^d$ or σ of $\mathbb{R}^{d-1}$ respectively.*

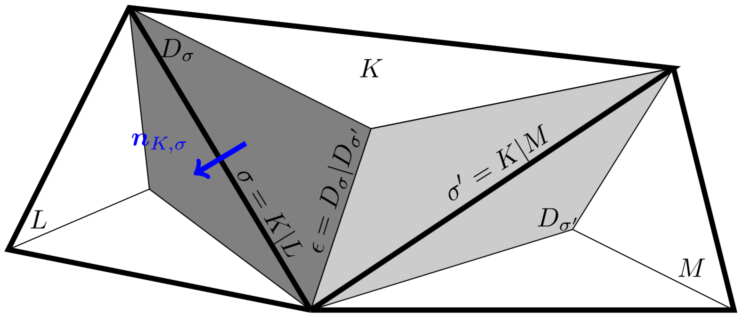

Figure 1: Notations for primal and dual cells. Primal cells are delimited with bold lines, dual cells are in grey.

- *We define a dual mesh associated with the faces $\sigma \in \mathcal{E}$ as follows. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, we define $D_{K,\sigma}$ as the cone with basis σ and with vertex the mass center of K (see Figure 1). We thus obtain a partition of K in m sub-volumes, where $m = d+1$ is the number of faces of K, each sub-volume having the same measure $|D_{K,\sigma}| = |K|/(d+1)$. The volume $D_{K,\sigma}$ is referred to as the half-diamond cell associated with K and σ. For $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, we now define the diamond cell D_σ associated with σ by $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$. For $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}(K)$, we define $D_\sigma = D_{K,\sigma}$. We denote by $\tilde{\mathcal{E}}(D_\sigma)$ the set of faces of D_σ, and by $\epsilon = D_\sigma|D_{\sigma'}$ the face separating two diamond cells D_σ and $D_{\sigma'}$. As for the primal mesh, we denote by $\tilde{\mathcal{E}}_{\rm int}$ the set of dual faces included in the domain Ω and by $\tilde{\mathcal{E}}_{\rm ext}$ the set of dual faces lying on the boundary $\partial\Omega$. In this latter case, there exists $\sigma \in \mathcal{E}_{\rm ext}$ such that $\epsilon = \sigma$.*

Definition 3.2 (Size of the discretization)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. For every $K \in \mathcal{M}$, we denote h_K the diameter of K (*i.e.* the 1D measure of the largest line segment included in K). The size of the discretization is defined by:*

$$h_{\mathcal{M}} = \max_{K \in \mathcal{M}} h_K.$$

Definition 3.3 (Regularity of the discretization)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. For every $K \in \mathcal{M}$, denote ϱ_K the radius of the largest ball included in K. The regularity parameter of the discretization is defined by:*

$$\theta_{\mathcal{M}} = \max\Big\{\frac{h_K}{\varrho_K},\, K \in \mathcal{M}\Big\} \cup \Big\{\frac{h_K}{h_L},\, \frac{h_L}{h_K},\, \sigma = K|L \in \mathcal{E}_{\rm int}\Big\}. \tag{3.1}$$

A sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of staggered discretizations is said to be regular if:

(i) there exists $\theta_0 > 0$ such that $\theta_{\mathcal{M}_n} \le \theta_0$ for all $n \in \mathbb{N}$,

(ii) the sequence of space steps $(h_{\mathcal{M}_n})_{n\in\mathbb{N}}$ tends to zero as n tends to $+\infty$.

Relying on Definition 3.1, we now define a staggered space discretization. The degrees of freedom for the density (*i.e.* the discrete density unknowns) are associated with the cells of the mesh $\mathcal{M}$:

$$\big\{\rho_K,\ K \in \mathcal{M}\big\}.$$

The discrete density unknowns are associated with piecewise constant functions on the cells of the primal mesh.

Definition 3.4. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. We denote $\mathrm{L}_{\mathcal{M}}(\Omega)$ the space of scalar functions that are constant on each primal cell $K \in \mathcal{M}$. For $\rho \in \mathrm{L}_{\mathcal{M}}(\Omega)$ and $K \in \mathcal{M}$, we denote ρ_K the constant value of ρ on K. We denote $\mathrm{L}_{\mathcal{M},0}(\Omega)$ the subspace of $\mathrm{L}_{\mathcal{M}}(\Omega)$ composed of zero average functions over Ω.*

The degrees of freedom for the velocity are associated with the faces of the mesh $\mathcal{M}$ or equivalently with the cells of the dual mesh D_σ, $\sigma \in \mathcal{E}$ so the set of discrete velocity unknowns reads:

$$\{\boldsymbol{u}_\sigma \in \mathbb{R}^d,\ \sigma \in \mathcal{E}\}.$$

The discrete velocity unknowns are associated with the *Crouzeix-Raviart* finite element. For all $K \in \mathcal{M}$, the restriction of the discrete velocity belongs to $P_1(K)$ the space of polynomials of degree less than one defined on K.

The space of discrete velocities is given in the following definition.

Definition 3.5. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω as defined in Definition 3.1. We denote $\mathrm{H}_{\mathcal{M}}(\Omega)$ the space of functions u such that $u_{|K} \in P_1(K)$ for all $K \in \mathcal{M}$ and such that:*

$$\frac{1}{|\sigma|}\int_\sigma [u]_\sigma \,\mathrm{d}\sigma(\boldsymbol{x}) = 0, \qquad \forall \sigma \in \mathcal{E}_{\mathrm{int}}, \tag{3.2}$$

where $[u]_\sigma$ is the jump of u through σ which is defined on $\sigma = K|L$ by $[u]_\sigma = u_{|L} - u_{|K}$. We define $\mathrm{H}_{\mathcal{M},0}(\Omega) \subset \mathrm{H}_{\mathcal{M}}(\Omega)$ the subspace of $\mathrm{H}_{\mathcal{M}}(\Omega)$ composed of functions the degrees of freedom of which are zero over $\partial\Omega$, i.e. *the functions $u \in \mathrm{H}_{\mathcal{M}}(\Omega)$ such that $\frac{1}{|\sigma|}\int_\sigma u \,\mathrm{d}\sigma(\boldsymbol{x}) = 0$ for all $\sigma \in \mathcal{E}_{\mathrm{ext}}$. Finally, we denote $\mathbf{H}_{\mathcal{M}}(\Omega) := \mathrm{H}_{\mathcal{M}}(\Omega)^d$ and $\mathbf{H}_{\mathcal{M},0}(\Omega) := \mathrm{H}_{\mathcal{M},0}(\Omega)^d$.*

For a discrete velocity field $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$ and $\sigma \in \mathcal{E}$, the degree of freedom associated with σ is given by:

$$\boldsymbol{u}_\sigma = \frac{1}{|\sigma|}\int_\sigma \boldsymbol{u} \,\mathrm{d}\sigma(\boldsymbol{x}). \tag{3.3}$$

Although $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$ is discontinuous across an internal face $\sigma \in \mathcal{E}_{\mathrm{int}}$, the definition of $\boldsymbol{u}_\sigma$ is unambiguous thanks to (3.2).

3.2 The numerical scheme

Let Ω be a polyhedral domain of $\mathbb{R}^d$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω as defined in Definition 3.1. The continuity equation is discretized on the primal mesh, while the momentum balance is discretized on the dual mesh. The scheme reads as follows:

Solve for $\rho \in \mathrm{L}_{\mathcal{M}}(\Omega)$ and $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$:

$$\mathrm{div}_{\mathcal{M}}(\rho\boldsymbol{u}) + h_{\mathcal{M}}^{\xi_1}(\rho - \rho^{\star}) - h_{\mathcal{M}}^{\xi_2}\,\Delta_{\frac{1+\eta}{\eta},\mathcal{M}}(\rho) = 0, \tag{3.4a}$$

$$\mathbf{div}_{\mathcal{E}}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \mu\boldsymbol{\Delta}_{\mathcal{E}}\boldsymbol{u} - (\mu+\lambda)(\boldsymbol{\nabla}\circ\mathrm{div})_{\mathcal{E}}\boldsymbol{u} + a\boldsymbol{\nabla}_{\mathcal{E}}(\rho^{\gamma}) + h_{\mathcal{M}}^{\xi_3}\,\boldsymbol{\nabla}_{\mathcal{E}}(\rho^{\Gamma}) = \widetilde{\Pi}_{\mathcal{E}}\boldsymbol{f}, \tag{3.4b}$$

where $\eta = \frac{2\gamma-3}{\gamma}$.

The discrete space differential operators involved in (3.4a) and (3.4b), as well as their main properties, are described in the following lines. The positive constants Γ and (ξ_1, ξ_2, ξ_3) will be determined so as to ensure the convergence of the numerical solution towards a weak solution of (1.1)-(1.2)-(1.3).

Mass convection operator – Given discrete density and velocity fields $\rho \in \mathrm{L}_{\mathcal{M}}(\Omega)$ and $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$, the discretization of the mass convection term is given by:

$$\mathrm{div}_{\mathcal{M}}(\rho\boldsymbol{u})(\boldsymbol{x}) = \sum_{K\in\mathcal{M}} \frac{1}{|K|}\Big(\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}(\rho,\boldsymbol{u})\Big)\,\mathcal{X}_K(\boldsymbol{x}), \tag{3.5}$$

where $\mathcal{X}_K$ is the characteristic function of the subset K of Ω. The quantity $F_{K,\sigma}(\rho,\boldsymbol{u})$ stands for the mass flux across σ outward K. By the impermeability boundary conditions, it vanishes on external faces and is given on internal faces by:

$$F_{K,\sigma}(\rho,\boldsymbol{u}) = |\sigma|\,\rho_\sigma\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}, \qquad \forall\sigma\in\mathcal{E}_{\mathrm{int}},\ \sigma = K|L. \tag{3.6}$$

The density at the face $\sigma = K|L$ is approximated by the upwind technique, *i.e.*

$$\rho_\sigma = \begin{cases} \rho_K & \text{if} \quad \boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma} \geq 0, \\ \rho_L & \text{otherwise.} \end{cases} \tag{3.7}$$

Stabilization terms in the mass equation – The discrete mass equation involves two stabilization terms. The first stabilization term is there to ensure the total mass constraint at the discrete level (1.3):

$$h_{\mathcal{M}}^{\xi_1}(\rho(\boldsymbol{x}) - \rho^{\star}) = h_{\mathcal{M}}^{\xi_1}\sum_{K\in\mathcal{M}}(\rho_K - \rho^{\star})\mathcal{X}_K(\boldsymbol{x}).$$

The second stabilization term in the discrete mass equation (3.4a) is defined as follows:

$$-\Delta_{\frac{1+\eta}{\eta},\mathcal{M}}(\rho)(\boldsymbol{x}) = \sum_{K\in\mathcal{M}}\frac{1}{|K|}\Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma|\left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}-1}(\rho_K-\rho_L)\Big)\,\mathcal{X}_K(\boldsymbol{x}). \tag{3.8}$$

Its aim is to provide a control on a discrete analogue of the $\mathrm{W}^{1,\frac{1+\eta}{\eta}}(\Omega)$ semi-norm of ρ by some (negative) power of the discretization parameter $h_{\mathcal{M}}$. This control appears to be necessary in the convergence analysis, when passing to the limit in the equation of state, see Remark 5.1.

Velocity convection operator – Given discrete density and velocity fields $\rho \in \mathrm{L}_{\mathcal{M}}(\Omega)$ and $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$, the discretization of the mass convection term is given by:

$$\mathbf{div}_{\mathcal{E}}(\rho \boldsymbol{u} \otimes \boldsymbol{u})(\boldsymbol{x}) = \sum_{\sigma \in \mathcal{E}_{\rm int}} \frac{1}{|D_\sigma|} \Big(\sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\rho, \boldsymbol{u})\ \boldsymbol{u}_\epsilon \Big)\, \mathcal{X}_{D_\sigma}(\boldsymbol{x}). \tag{3.9}$$

$F_{\sigma,\epsilon}(\rho, \boldsymbol{u})$ is the mass flux across the edge ϵ of the dual cell D_σ. Its value is zero if $\epsilon \in \tilde{\mathcal{E}}_{\rm ext}$. Otherwise, it is defined as a linear combination, with constant coefficients, of the primal mass fluxes at the neighboring faces. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, let ξ_K^σ be given by:

$$\xi_K^\sigma = \frac{|D_{K,\sigma}|}{|K|} = \frac{1}{d+1}, \tag{3.10}$$

so that $\sum_{\sigma \in \mathcal{E}(K)} \xi_K^\sigma = 1$. Then the mass fluxes through the inner dual faces are calculated from the primal mass fluxes $F_{K,\sigma}(\rho, \boldsymbol{u})$ as follows. We first incorporate the second stabilization term (see (3.8)) into the primal mass fluxes by defining $\overline{F}_{K,\sigma}(\rho, \boldsymbol{u})$ as follows:

$$\overline{F}_{K,\sigma}(\rho, \boldsymbol{u}) = F_{K,\sigma}(\rho, \boldsymbol{u}) + h_{\mathcal{M}}^{\xi_2} |\sigma| \Big(\frac{|\sigma|}{|D_\sigma|} \Big)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L), \quad \forall \sigma \in \mathcal{E}_{\rm int},\, \sigma = K|L. \tag{3.11}$$

The dual mass fluxes $F_{\sigma,\epsilon}(\rho, \boldsymbol{u})$ are then computed to as to satisfy the following three conditions:

(H1) The discrete mass balance over the half-diamond cells is satisfied, in the following sense. For all primal cell K in $\mathcal{M}$, the set $(F_{\sigma,\epsilon}(\rho, \boldsymbol{u}))_{\epsilon \subset K}$ of dual fluxes included in K solves the following linear system

$$\overline{F}_{K,\sigma}(\rho, \boldsymbol{u}) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma),\ \epsilon \subset K} F_{\sigma,\epsilon}(\rho, \boldsymbol{u}) = \xi_K^\sigma \sum_{\sigma' \in \mathcal{E}(K)} \overline{F}_{K,\sigma'}(\rho, \boldsymbol{u}), \quad \sigma \in \mathcal{E}(K). \tag{3.12}$$

(H2) The dual fluxes are conservative, *i.e.*

$$F_{\sigma,\epsilon}(\rho, \boldsymbol{u}) = -F_{\sigma',\epsilon}(\rho, \boldsymbol{u}), \qquad \forall \epsilon = D_\sigma | D_{\sigma'}. \tag{3.13}$$

(H3) The dual fluxes are bounded with respect to the primal fluxes $(\overline{F}_{K,\sigma}(\rho, \boldsymbol{u}))_{\sigma \in \mathcal{E}(K)}$, in the sense that

$$|F_{\sigma,\epsilon}(\rho, \boldsymbol{u})| \leq \ \max \left\{ |\overline{F}_{K,\sigma'}(\rho, \boldsymbol{u})|,\ \sigma' \in \mathcal{E}(K) \right\}, \tag{3.14}$$

for $K \in \mathcal{M}$, $\sigma \in \mathcal{E}(K)$, $\epsilon \in \tilde{\mathcal{E}}(D_\sigma)$ with $\epsilon \subset K$.

The system of equations (3.12)-(3.13) does not depend on the particular cell K since it only depends on the coefficient $\xi_K^\sigma = 1/(d+1)$. It has an infinite number of solutions, which makes necessary to impose in addition the constraint (3.14); however, assumptions (H1)-(H2)-(H3) are sufficient for the subsequent developments, in the sense that any choice for the expression of the dual fluxes satisfying these assumptions yields stable and consistent schemes (see [31, 32]).

This convection operator is built so that a discrete mass conservation equation similar to (3.4a) is also satisfied on the cells of the dual mesh. Indeed, let $(\rho, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M}}(\Omega)$ and define a constant density on the dual cells ρ_{D_σ} as follows:

$$|D_\sigma| \rho_{D_\sigma} = |D_{K,\sigma}| \rho_K + |D_{L,\sigma}| \rho_L \quad \text{for } \sigma = K|L.$$

Then if $(\rho, \boldsymbol{u})$ satisfy (3.4a), one has:

$$\sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\rho, \boldsymbol{u}) + h_{\mathcal{M}}^{\xi_1} |D_\sigma| \left(\rho_{D_\sigma} - \rho^\star\right) = 0, \qquad \forall \sigma \in \mathcal{E}_{\mathrm{int}}, \tag{3.15}$$

which is an analogue of (3.4a) where the stabilization diffusion term is hidden in the dual fluxes.

To complete the definition of the momentum convective term, we must give the expression of the velocity $\boldsymbol{u}_\epsilon$ at the dual face. As already said, a dual face lying on the boundary is also a primal face, and the flux across that face is zero. Therefore, the values $\boldsymbol{u}_\epsilon$ are only needed at the internal dual faces; we choose them to be centered:

$$\boldsymbol{u}_\epsilon = \frac{1}{2}(\boldsymbol{u}_\sigma + \boldsymbol{u}_{\sigma'}), \qquad \text{for } \epsilon = D_\sigma | D'_\sigma.$$

Diffusion operator – Let us define the shape functions associated with the Crouzeix-Raviart finite element. These are the functions $(\zeta_\sigma)_{\sigma \in \mathcal{E}}$ where for all $\sigma \in \mathcal{E}$, ζ_σ is the element of $\mathrm{H}_{\mathcal{M}}(\Omega)$ which satisfies:

$$\frac{1}{|\sigma'|} \int_{\sigma'} \zeta_\sigma \, \mathrm{d}\sigma(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \sigma' = \sigma, \\ 0, & \text{if } \sigma' \neq \sigma. \end{cases}$$

Given a discrete velocity field $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$, the discretization of the diffusion terms is given by:

$$\begin{aligned} -\boldsymbol{\Delta}_{\mathcal{E}} \boldsymbol{u}(\boldsymbol{x}) &= \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \frac{1}{|D_\sigma|} \Big(\sum_{K \in \mathcal{M}} \int_K \boldsymbol{\nabla} \boldsymbol{u} \,.\, \boldsymbol{\nabla} \zeta_\sigma \, \mathrm{d}\boldsymbol{x} \Big) \mathcal{X}_{D_\sigma}(\boldsymbol{x}), \\ -(\boldsymbol{\nabla} \circ \mathrm{div})_{\mathcal{E}} \boldsymbol{u}(\boldsymbol{x}) &= \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \frac{1}{|D_\sigma|} \Big(\sum_{K \in \mathcal{M}} \int_K \mathrm{div}\, \boldsymbol{u} \boldsymbol{\nabla} \zeta_\sigma \, \mathrm{d}\boldsymbol{x} \Big) \mathcal{X}_{D_\sigma}(\boldsymbol{x}). \end{aligned} \tag{3.16}$$

Pressure gradient operator – Given a discrete density field $\rho \in \mathrm{L}_{\mathcal{M}}(\Omega)$, the pressure gradient term is discretized as follows:

$$\boldsymbol{\nabla}_{\mathcal{E}}(\rho^\gamma)(\boldsymbol{x}) = \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} \Big(\frac{|\sigma|}{|D_\sigma|} \, (\rho_L^\gamma - \rho_K^\gamma) \, \boldsymbol{n}_{K,\sigma} \Big) \mathcal{X}_{D_\sigma}(\boldsymbol{x}). \tag{3.17}$$

The discrete momentum equation (3.4b) also involves a third stabilization term, an artificial pressure term, which reads:

$$h_{\mathcal{M}}^{\xi_3} \, \boldsymbol{\nabla}_{\mathcal{E}}(\rho^\Gamma)(\boldsymbol{x}) = h_{\mathcal{M}}^{\xi_3} \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} \Big(\frac{|\sigma|}{|D_\sigma|} \, (\rho_L^\Gamma - \rho_K^\Gamma) \, \boldsymbol{n}_{K,\sigma} \Big) \mathcal{X}_{D_\sigma}(\boldsymbol{x}),$$

where $\Gamma > \gamma$ is chosen large enough to ensure a control on the discrete weak formulation of the convective term in the momentum equation when $\gamma \in (\frac{3}{2}, 3]$. Note that, if $d = 3$ and $\gamma > 3$ or $d = 2$ and $\gamma > 2$, this term is not needed.

Source term – The source term $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ is discretized with the following projection operator:

$$\widetilde{\Pi}_{\mathcal{E}} \boldsymbol{f}(\boldsymbol{x}) = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \Big(\frac{1}{|D_\sigma|} \int_{D_\sigma} \boldsymbol{f} \,\mathrm{d}\boldsymbol{x} \Big) \mathcal{X}_{D_\sigma}(\boldsymbol{x}). \tag{3.18}$$

3.3 Main result: convergence of the scheme

For the clarity of the presentation, we state our convergence result in the same setting as for the continuous problem, namely for $d = 3$ and $\gamma \in (\frac{3}{2}, 3]$. We refer to the remark below for the "simpler" cases $d = 2$, and $d = 3$ with $\gamma > 3$.

Theorem 3.1 (Convergence of the scheme)**.** *Let Ω be a polyhedral connected open subset of $\mathbb{R}^3$. Let $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ and $\rho^\star > 0$. Assume that $\gamma \in (\frac{3}{2}, 3]$. Denoting $\eta = \frac{2\gamma - 3}{\gamma} \in (0, 1]$, assume that Γ and (ξ_1, ξ_2, ξ_3) satisfy:*

$$(i) \qquad \xi_1 > 1, \tag{3.19}$$

$$(ii) \qquad \frac{5}{4\Gamma}\left(\frac{3}{1+\eta} + \xi_3 \right) < \frac{\eta}{1+\eta}, \tag{3.20}$$

$$(iii) \qquad \frac{1}{\eta} + \frac{5}{4\eta\Gamma}\left(\frac{3}{1+\eta} + \xi_3 \right) < \xi_2 < \frac{1+\eta}{\eta} - \frac{5}{4\Gamma}\left(\frac{3}{1+\eta} + \xi_3 \right). \tag{3.21}$$

Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a regular sequence of staggered discretizations of Ω as defined in Definition 3.3. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there exists a solution $(\rho_n, \boldsymbol{u}_n) \in \mathrm{L}_{\mathcal{M}_n}(\Omega) \times \mathbf{H}_{\mathcal{M}_n,0}(\Omega)$ to the numerical scheme (3.4) with the discretization $\mathcal{D}_n$ and the obtained density ρ_n is positive on Ω. Moreover, there exist $(\rho, \boldsymbol{u}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}_0^1(\Omega)$ and a subsequence of $(\rho_n, \boldsymbol{u}_n)_{n\geq N}$, denoted $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ such that:

- *The sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{u}$ in $\mathbf{L}^q(\Omega)$ for all $q \in [1, 6)$,*
- *The sequence $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ in $\mathrm{L}^q(\Omega)$ for all $q \in [1, 3(\gamma-1))$ and weakly in $\mathrm{L}^{3(\gamma-1)}(\Omega)$,*
- *The sequence $(\rho_n^\gamma)_{n\in\mathbb{N}}$ converges to ρ^γ in $\mathrm{L}^q(\Omega)$ for all $q \in [1, \frac{3(\gamma-1)}{\gamma})$ and weakly in $\mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$,*
- *The pair $(\rho, \boldsymbol{u})$ is a weak solution of Problem (1.1)-(1.2)-(1.3) with finite energy.*

Remark 3.1 (Some remarks on Theorem 3.1)**.**

- *Let us mention that the convergence result of Theorem 3.1 can be extended to the cases $d = 3$, $\gamma > 3$ and $d = 2$, $\gamma > 2$ with the mass stabilization term defined as*

$$-h_{\mathcal{M}}^{\xi_2} \Delta_{\mathcal{M}} \rho(\boldsymbol{x}) = h_{\mathcal{M}}^{\xi_2} \sum_{K\in\mathcal{M}} \frac{1}{|K|} \Big(\sum_{\substack{\sigma \in \mathcal{E}(K)\cap\mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} |\sigma| \frac{|\sigma|}{|D_\sigma|} (\rho_K - \rho_L) \Big) \mathcal{X}_K(\boldsymbol{x}),$$

and without the artificial pressure term. The required constraints on (ξ_1, ξ_2) *are the following:*

$$\xi_1 > 1 \text{ if } d = 3 \qquad \text{and} \qquad \xi_1 > 0 \text{ if } d = 2,$$
$$\frac{3}{2} < \xi_2 < 2.$$

In the case $d = 2$, $1 < \gamma \leq 2$, *we expect a convergence result with the stabilization term proposed in [13] combined with an artificial pressure term.*

- *The upper bound on* ξ_2 *is required when passing to the limit in the effective viscous flux at the discrete level (see Subsection 5.3.1 and (5.28)). The lower bound on* ξ_2 *is required for the control on the momentum convective term when deriving the discrete estimate on the density (see (4.21)), which explains why this constraint was not introduced in [13] for the Stokes equations.*

The following sections are devoted to the proof of Theorem 3.1. In Section 4.1, we introduce some notations and properties of the discretization. In Sections 4.2 to 4.5, we derive *a priori* estimates on the solution of the scheme and prove its existence provided a small enough space step $h_{\mathcal{M}}$. Finally, in Section 5, we prove Theorem 3.1 by successively passing to the limit in the discrete mass and momentum equations, and then in the equation of state.

4 Mesh independent estimates and existence of a discrete solution

4.1 Discrete norms and properties

We gather in this section some preliminary mathematical results which are useful for the analysis of the scheme. Similar results have been previously used by Gallouët *et al.* in their study [20] which also relies on a mixed FV-FE discretization. The interested reader is also referred to the books [9], [11], [8] and to the appendix of [23].

We start with defining the piecewise smooth first order derivative operators associated with the Crouzeix-Raviart non-conforming finite element representation of velocities $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$:

$$\boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{u}(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} \boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{x}) \mathcal{X}_K(\boldsymbol{x}), \tag{4.1}$$

$$\operatorname{div}_{\mathcal{M}} \boldsymbol{u}(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} \operatorname{div} \boldsymbol{u}(\boldsymbol{x}) \mathcal{X}_K(\boldsymbol{x}), \tag{4.2}$$

$$\operatorname{curl}_{\mathcal{M}} \boldsymbol{u}(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} \operatorname{curl} \boldsymbol{u}(\boldsymbol{x}) \mathcal{X}_K(\boldsymbol{x}). \tag{4.3}$$

Note that on each element $K \in \mathcal{M}$, $\boldsymbol{\nabla} \boldsymbol{u}_{|K} \in \mathbb{R}^d$ is actually a constant and the divergence defined in (4.2) matches the finite volume divergence defined in (3.5) for $\rho \equiv 1$.

We then define for $q \in [1, \infty)$ the broken Sobolev $\mathrm{W}^{1,q}$ semi-norm $\|.\|_{1,q,\mathcal{M}}$ associated with the Crouzeix-Raviart finite element representation of the discrete velocities. For any $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$ it is given by:

$$\|\boldsymbol{u}\|_{1,q,\mathcal{M}}^q = \int_\Omega |\boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{u}|^q \, \mathrm{d}\boldsymbol{x}.$$

A first important property, proved in Appendix C, is the following discrete Sobolev embedding.

Lemma 4.1 (Discrete Sobolev embedding)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. Then, for all $q \in [1, +\infty)$ if $d = 2$ and for all $q \in [1, 6]$ if $d = 3$, there exists $C = C(q, d, \theta_0) > 0$ such that:*

$$\|\boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \le C \, \|\boldsymbol{u}\|_{1,2,\mathcal{M}}, \quad \forall \boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega).$$

Remark 4.1. *For d=2, one has $C(q, d, \theta_0) \to \infty$ as $q \to \infty$.*

A consequence of this Sobolev embedding is a discrete Poincaré inequality. Note that the semi-norm $\|\boldsymbol{u}\|_{1,2,\mathcal{M}}$ is in fact a norm on the space $\mathbf{H}_{\mathcal{M},0}(\Omega)$.

Lemma 4.2 (Discrete Poincaré inequality)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. Then there exists $C = C(d, \theta_0)$ such that*

$$\|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \le C \, \|\boldsymbol{u}\|_{1,2,\mathcal{M}}, \quad \forall \boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega).$$

It will be convenient in the analysis of the scheme to handle several representations of the discrete velocities. We define an interpolation operator $\Pi_{\mathcal{E}}$ which associates a piecewise constant function over the cells of the dual mesh to any function $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$ as follows:

$$\Pi_{\mathcal{E}} \boldsymbol{u}(\boldsymbol{x}) = \sum_{\sigma \in \mathcal{E}_{\rm int}} \boldsymbol{u}_\sigma \, \mathcal{X}_{D_\sigma}(\boldsymbol{x}). \tag{4.4}$$

The constant value of $\Pi_{\mathcal{E}} \boldsymbol{u}$ over the cell D_σ is $\boldsymbol{u}_\sigma$ defined in (3.3). The mapping $\boldsymbol{u} \mapsto \Pi_{\mathcal{E}} \boldsymbol{u}$ is a one-to-one mapping which is continuous with respect to the L^q-norm, for all $q \in [1, +\infty]$. Indeed, we have the following result.

Lemma 4.3. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. Then, for all $q \in [1, +\infty]$, there exists a constant $C = C(q, d, \theta_0)$ such that:*

$$\|\Pi_{\mathcal{E}} \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \le C \, \|\boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}.$$

Proof. This property is obtained through standard properties of the affine mapping and invoking a norm equivalence argument for the finite dimensional polynomial space on the reference unit simplex $P_1(\hat{K})$. □

We also define a finite-volume type gradient for the velocities associated with the dual mesh. This gradient is somehow a vector version of the gradient $\boldsymbol{\nabla}_{\mathcal{E}}$ defined in (3.17) for scalar function in $\mathrm{L}_{\mathcal{M}}(\Omega)$. For $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$ and $K \in \mathcal{M}$, denote $\boldsymbol{u}_K = \sum_{\sigma \in \mathcal{E}(K)} \xi_K^\sigma \, \boldsymbol{u}_\sigma$ where ξ_K^σ is defined in (3.10). The finite-volume gradient of $\boldsymbol{u}$ is defined by:

$$\boldsymbol{\nabla}_{\mathcal{E}} \boldsymbol{u}(\boldsymbol{x}) = \sum_{\substack{\sigma \in \mathcal{E}_{\rm int} \\ \sigma = K|L}} \Big(\frac{|\sigma|}{|D_\sigma|} (\boldsymbol{u}_L - \boldsymbol{u}_K) \otimes \boldsymbol{n}_{K,\sigma} \Big) \mathcal{X}_{D_\sigma}(\boldsymbol{x}). \tag{4.5}$$

We also introduce the following other discrete $\mathrm{W}^{1,q}$ semi-norm given for $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$ by:

$$\|\boldsymbol{u}\|_{1,q,\mathcal{E}}^q = \sum_{K \in \mathcal{M}} h_K^{d-q} \sum_{\sigma,\sigma' \in \mathcal{E}(K)} |\boldsymbol{u}_\sigma - \boldsymbol{u}_{\sigma'}|^q.$$

This semi-norm may be shown to be equivalent, over a regular sequence of discretizations, to the usual finite volume $\mathrm{W}^{1,q}$ semi-norm associated with the piecewise constant function $\Pi_{\mathcal{E}}\boldsymbol{u}$. It is possible to prove that this semi-norm, as well as the semi-norm defined by the L^q norm of $\boldsymbol{\nabla}_{\mathcal{E}}\boldsymbol{u}$ are controlled on a regular discretization by the finite-element $\mathrm{W}^{1,q}$ semi-norm. Indeed, we have the following lemma.

Lemma 4.4. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \leq \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. Then for all $q \in [1, +\infty)$ there exist two constants $C_1 = C_1(q, d, \theta_0)$ and $C_2 = C_2(q, d, \theta_0)$ such that:*

$$\|\boldsymbol{\nabla}_{\mathcal{E}}\boldsymbol{u}\|_{\mathbf{L}^q(\Omega)^d} \leq C_1 \|\boldsymbol{u}\|_{1,q,\mathcal{E}} \leq C_2 \|\boldsymbol{u}\|_{1,q,\mathcal{M}}, \quad \forall \boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega).$$

Proof. The first inequality follows from the regularity of the discretization and from the fact that for $K \in \mathcal{M}$, $\boldsymbol{u}_K$ is a convex combination of $(\boldsymbol{u}_\sigma)_{\sigma \in \mathcal{E}(K)}$. The second inequality is obtained through standard properties of the affine mapping and invoking a norm equivalence argument for the finite dimensional polynomial space on the reference unit simplex $P_1(\hat{K})$. □

Lemma 4.5 (Inverse inequalities)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \leq \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. Let u be a function defined on Ω such that for all $K \in \mathcal{M}$, $u_{|K}$ belongs to a finite dimensional space of functions which is stable by affine transformation. Then, for all $q, p \in [1, +\infty]$, there exists $C = C(q, p, \theta_0, d)$ such that (with $1/\infty = 0$):*

$$\|u\|_{\mathrm{L}^q(K)} \leq C\, h_K^{d\left(\frac{1}{q} - \frac{1}{p}\right)} \|u\|_{\mathrm{L}^p(K)}, \qquad \forall K \in \mathcal{M}. \tag{4.6}$$

Hence, for all $p \in [1, +\infty)$, there exists $C = C(p, \theta_0)$ such that :

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \leq C\, h_{\mathcal{M}}^{-\frac{d}{p}} \|u\|_{\mathrm{L}^p(\Omega)}. \tag{4.7}$$

Proof. Inequality (4.7) is a direct consequence of (4.6). Let us prove the latter. Let $\hat{K}$ be the reference element. We have $K = \mathcal{A}_K(\hat{K})$ for some affine mapping $\mathcal{A}_K$. We denote $\hat{u}$ the function defined by $\hat{u}(\hat{\boldsymbol{x}}) = u(\boldsymbol{x})$ for $\boldsymbol{x} = \mathcal{A}_K(\hat{\boldsymbol{x}})$. By this change of variable we have:

$$\left(\frac{|\hat{K}|}{|K|}\right)^{-\frac{1}{q}} \|u\|_{\mathrm{L}^q(K)} \leq \|\hat{u}\|_{\mathrm{L}^q(\hat{K})}, \qquad \|\hat{u}\|_{\mathrm{L}^p(\hat{K})} \leq \left(\frac{|\hat{K}|}{|K|}\right)^{\frac{1}{p}} \|u\|_{\mathrm{L}^p(K)}.$$

Since $\hat{u}_{|\hat{K}}$ belongs to a finite dimensional space, the equivalence of all norms gives $\|\hat{u}\|_{\mathrm{L}^q(\hat{K})} \leq C \|\hat{u}\|_{\mathrm{L}^p(\hat{K})}$ with $C = C(q, p, d)$. We conclude by invoking the regularity of the discretization: $C(\theta_0)^{-1} h_K^d \leq |K| \leq C(\theta_0)\, h_K^d$.

□

For $q \in [1, +\infty)$, we introduce a discrete semi-norm on $\mathrm{L}_{\mathcal{M}}(\Omega)$ similar to the usual $\mathrm{W}^{1,q}$ semi-norm used in the finite volume context:

$$|\rho|^q_{q,\mathcal{M}} := \|\boldsymbol{\nabla}_{\mathcal{E}}(\rho)\|^q_{\mathbf{L}^q(\Omega)} = \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} |D_\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^q |\rho_K - \rho_L|^q.$$

It will be convenient in the analysis of the scheme to handle another representation of the discrete densities associated with the upwind discretization of the mass flux. For $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$ we define an interpolation operator $\mathcal{P}_{\mathcal{E}}$ which associates a piecewise constant function over the cells of the dual mesh to any function $\rho \in \mathrm{L}_{\mathcal{M}}(\Omega)$ as follows:

$$\mathcal{P}_{\mathcal{E}}\rho(\boldsymbol{x}) = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \rho_\sigma\, \mathcal{X}_{D_\sigma}(\boldsymbol{x}). \tag{4.8}$$

The constant value of $\mathcal{P}_{\mathcal{E}}\rho$ over the cell D_σ, $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ is ρ_σ the upwind value with respect to $\boldsymbol{u}_\sigma$, *i.e.* $\rho_\sigma = \rho_K$ is $\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} \geq 0$ and $\rho_\sigma = \rho_L$ otherwise.

Lemma 4.6. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \leq \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. For all $q \in [1, +\infty]$, there exists a constant $C = C(q, \theta_0)$ such that:*

$$\|\mathcal{P}_{\mathcal{E}}\rho\|_{\mathrm{L}^q(\Omega)} \leq C\|\rho\|_{\mathrm{L}^q(\Omega)}, \quad \forall \rho \in \mathrm{L}_{\mathcal{M}}(\Omega).$$

Proof. The proof is trivial for $q = +\infty$. Let $q \in [1, +\infty)$, in that case we have:

$$\|\mathcal{P}_{\mathcal{E}}\rho\|^q_{\mathrm{L}^q(\Omega)} = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |D_\sigma|\, |\rho_\sigma|^q.$$

For $\sigma = K|L$, either $\rho_\sigma = \rho_K$ or $\rho_\sigma = \rho_L$. Hence, $|D_\sigma|\,|\rho_\sigma|^q \leq |D_\sigma|\,|\rho_K|^q + |D_\sigma|\,|\rho_L|^q$. Thanks to the regularity of the mesh, there exists $C(\theta_0)$ such that $|D_\sigma|\,|\rho_\sigma|^q \leq C(\theta_0)(|D_{K,\sigma}|\,|\rho_K|^q + |D_{L,\sigma}|\,|\rho_L|^q)$. Summing the right hand side of this inequality over $\sigma \in \mathcal{E}_{\mathrm{int}}$ yields $C(\theta_0)\,\|\rho\|^q_{\mathrm{L}^q(\Omega)}$. □

In the following subsections, the discrete density is first shown to be positive and we prove a discrete analogue of the renormalization property satisfied by the solutions of the discrete mass equation (3.4a). Then, we establish stability properties enjoyed by any solution of the numerical scheme which are discrete analogues of the uniform (with respect to n) estimates of Section 2 for the solutions of the continuous problem. In particular, we prove a discrete $\mathbf{H}^1_0$-estimate on the velocity and a the control of the density in $\mathrm{L}^{3(\gamma-1)}(\Omega)$ which are both independent of the mesh size $h_{\mathcal{M}}$. In order to perform the convergence analysis of the scheme, we also prove stronger controls on the density (an L^Γ-control with Γ large and a control on the discrete gradient) which nevertheless blow up as the mesh size $h_{\mathcal{M}}$ tends to zero. From this point on, we assume $d = 3$.

4.2 Positivity of the density and discrete renormalization property

We prove the positivity of the discrete density $\rho \in \mathrm{L}_{\mathcal{M}}(\Omega)$ if $(\rho, \boldsymbol{u})$ is a solution of the discrete mass balance (3.4a). We have the following result:

Proposition 4.7 (Positivity of the density)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. Let $(\rho, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M}}(\Omega)$ be a solution of the discrete mass balance (3.4a). For $K \in \mathcal{M}$, denote* $\mathrm{div}(\boldsymbol{u})_K$ *the constant value of* $\mathrm{div}_{\mathcal{M}}\, \boldsymbol{u}$ *over K. Then*

$$\rho_K \geq \bar{\rho} := \frac{\rho^\star}{1 + h_{\mathcal{M}}^{-\xi_1} \max\left(0, \max\limits_{K\in\mathcal{M}} \mathrm{div}(\boldsymbol{u})_K\right)} > 0 \quad \forall K \in \mathcal{M}.$$

Proof. We proceed by contradiction. Let $\bar{K} \in \mathcal{M}$ be such that $\rho_{\bar{K}} = \min\limits_{K\in\mathcal{M}} \rho_K$ and assume that $\rho_{\bar{K}} < \bar{\rho}$. Multiplying (3.4a) by $\mathcal{X}_{\bar{K}}$ we get:

$$h_{\mathcal{M}}^{\xi_1} (\rho_{\bar{K}} - \rho^\star) + \rho_{\bar{K}}\, \mathrm{div}(\boldsymbol{u})_{\bar{K}} + \frac{1}{|\bar{K}|} \sum_{\sigma\in\mathcal{E}(\bar{K})} |\sigma|\, (\rho_\sigma - \rho_{\bar{K}})\, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{\bar{K},\sigma} \\ + h_{\mathcal{M}}^{\xi_2} \frac{1}{|\bar{K}|} \sum_{\substack{\sigma\in\mathcal{E}(\bar{K})\\ \sigma=\bar{K}|L}} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_{\bar{K}} - \rho_L|^{\frac{1}{\eta}-1} (\rho_{\bar{K}} - \rho_L) = 0.$$

For L a neighboring cell to $\bar{K}$, one has $\rho_L \geq \rho_{\bar{K}}$ by definition of $\bar{K}$, which implies that the last term is non-positive. In addition, because of the upwind definition of ρ_σ (see (3.7)), the third term is also non-positive. Hence, we get:

$$h_{\mathcal{M}}^{\xi_1} (\rho_{\bar{K}} - \rho^\star) + \rho_{\bar{K}}\, \mathrm{div}(\boldsymbol{u})_{\bar{K}} \geq 0. \tag{4.9}$$

By definition of $\bar{\rho}$, we have:

$$h_{\mathcal{M}}^{\xi_1} (\bar{\rho} - \rho^\star) + \bar{\rho} \max\left(0, \max_{K\in\mathcal{M}} \mathrm{div}(\boldsymbol{u})_K\right) = 0. \tag{4.10}$$

Subtracting (4.10) from (4.9) we obtain:

$$h_{\mathcal{M}}^{\xi_1} (\rho_{\bar{K}} - \bar{\rho}) + \left(\rho_{\bar{K}}\, \mathrm{div}(\boldsymbol{u})_{\bar{K}} - \bar{\rho} \max\left(0, \max_{K\in\mathcal{M}} \mathrm{div}(\boldsymbol{u})_K\right)\right) \geq 0.$$

If $\mathrm{div}(\boldsymbol{u})_{\bar{K}} \leq 0$ then the second term is clearly non-positive. If $\mathrm{div}(\boldsymbol{u})_{\bar{K}} \geq 0$, then we have

$$\begin{aligned} \rho_{\bar{K}}\, \mathrm{div}(\boldsymbol{u})_{\bar{K}} - \bar{\rho} \max\left(0, \max_{K\in\mathcal{M}} \mathrm{div}(\boldsymbol{u})_K\right) &= \rho_{\bar{K}}\, \mathrm{div}(\boldsymbol{u})_{\bar{K}} - \bar{\rho} \max_{K\in\mathcal{M}} \mathrm{div}(\boldsymbol{u})_K \\ &\leq \bar{\rho}\Big(\mathrm{div}(\boldsymbol{u})_{\bar{K}} - \max_{K\in\mathcal{M}} \mathrm{div}(\boldsymbol{u})_K\Big) \end{aligned}$$

which is also non-positive. In both cases we obtain that $h_{\mathcal{M}}^{\xi_1} (\rho_{\bar{K}} - \bar{\rho}) \geq 0$ which contradicts the assumption $\rho_{\bar{K}} < \bar{\rho}$.

□

Next, we state the following result which is a discrete analogue of the renormalization property (2.6) satisfied at the continuous level. The proof is given in Appendix A.1.

Proposition 4.8 (Discrete renormalization property)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. Let $(\rho, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M},0}(\Omega)$ satisfy the discrete mass balance* (3.4a)*. We have $\rho > 0$* a.e. *in Ω (*i.e. *$\rho_K > 0$, $\forall K \in \mathcal{M}$). Then, for any $b \in \mathcal{C}^1([0, +\infty))$:*

$$\mathrm{div}\big(b(\rho)\boldsymbol{u}\big)_K + \big(b'(\rho_K)\rho_K - b(\rho_K)\big)\mathrm{div}(\boldsymbol{u})_K + R_K^1 + R_K^2 + R_K^3 = 0 \quad \forall K \in \mathcal{M}, \tag{4.11}$$

where

$$\mathrm{div}\big(b(\rho)\boldsymbol{u}\big)_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \; b(\rho_\sigma)\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma},$$

and

$$R_K^1 = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| r_{K,\sigma} \boldsymbol{u}_\sigma \cdot n_{K,\sigma} \quad \textit{and} \quad r_{K,\sigma} = b'(\rho_K)(\rho_\sigma - \rho_K) + b(\rho_K) - b(\rho_\sigma),$$

$$R_K^2 = h_{\mathcal{M}}^{\xi_2} \; b'(\rho_K) \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L),$$

$$R_K^3 = h_{\mathcal{M}}^{\xi_1} b'(\rho_K)(\rho_K - \rho^\star).$$

Multiplying by $|K|$ and summing over $K \in \mathcal{M}$, it holds

$$\int_\Omega \big(b'(\rho)\rho - b(\rho)\big)\mathrm{div}_{\mathcal{M}} \, \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} + R_{\mathcal{E}}^1 + R_{\mathcal{E}}^2 + R_{\mathcal{M}}^3 = 0, \tag{4.12}$$

with

$$R_{\mathcal{E}}^1 = \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} |\sigma| (r_{K,\sigma} - r_{L,\sigma}) \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma},$$

$$R_{\mathcal{E}}^2 = h_{\mathcal{M}}^{\xi_2} \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L)(b'(\rho_K) - b'(\rho_L)),$$

$$R_{\mathcal{M}}^3 = h_{\mathcal{M}}^{\xi_1} \sum_{K \in \mathcal{M}} |K| b'(\rho_K)(\rho_K - \rho^\star),$$

and if b is convex then $R_{\mathcal{E}}^{1,2} \geq 0$ and $R_{\mathcal{M}}^3 \geq 0$.

Remark 4.2. *For $b(\rho) = \dfrac{1}{\beta - 1}\rho^\beta$ with $\beta > 1$, the previous result gives*

$$\int_\Omega \rho^\beta \, \mathrm{div}_{\mathcal{M}} \, \boldsymbol{u} \; \mathrm{d}\boldsymbol{x} + R_{\mathcal{E},1}(\rho, \boldsymbol{u}) + R_{\mathcal{E},2}(\rho, \boldsymbol{u}) \leq 0, \tag{4.13}$$

with

$$R_{\mathcal{E}}^1(\rho, \boldsymbol{u}) = \frac{\beta}{2} \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} |\sigma| \; \min(\rho_K^{\beta-2}, \rho_L^{\beta-2}) \, (\rho_L - \rho_K)^2 \, |\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}| \geq 0,$$

$$R_{\mathcal{E}}^2(\rho, \boldsymbol{u}) = h_{\mathcal{M}}^{\xi_2} \frac{\beta}{\beta - 1} \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L)(\rho_K^{\beta-1} - \rho_L^{\beta-1}) \geq 0.$$

Remark 4.3. *As explained in the continuous case, we can extend the renormalization result of Proposition 4.8 to functions $b \in \mathcal{C}^0([0,+\infty)) \cap \mathcal{C}^1((0,+\infty))$, the derivative of which is not bounded close to 0, but satisfies for all $t \leq 1$:*

$$|b'(t)| \leq Ct^{-\lambda_0} \quad \textit{for some } \lambda_0 < 1.$$

Remark 4.4. *The positivity of the density and the discrete renormalization property are actually independent of the space dimension. They are naturally valid for $d = 2$.*

4.3 Estimate on the discrete velocity

In order to derive estimates on the discrete velocity and density, we proceed as in the continuous case and begin with writing a discrete counterpart of the weak formulation of the momentum balance. We begin with the following Lemma which states discrete counterparts to classical Stokes formulas. We refer to Sections 3.2 and 4.1 for the definitions of the operators.

Lemma 4.9. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. The following discrete integration by parts formulas are satisfied for all $(p, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M},0}(\Omega)$. One has for all $\boldsymbol{v} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$:*

$$-\int_\Omega \boldsymbol{\Delta}_{\mathcal{E}} \boldsymbol{u} \cdot \Pi_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{u} : \boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \tag{4.14}$$

$$-\int_\Omega (\boldsymbol{\nabla} \circ \mathrm{div})_{\mathcal{E}} \boldsymbol{u} \cdot \Pi_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div}_{\mathcal{M}}\, \boldsymbol{u} \,\mathrm{div}_{\mathcal{M}}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \tag{4.15}$$

$$\int_\Omega \boldsymbol{\nabla}_{\mathcal{E}}(p) \cdot \Pi_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = -\int_\Omega p \,\mathrm{div}_{\mathcal{M}}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}. \tag{4.16}$$

Proof. To exemplify, we prove (4.14). We have:

$$\begin{aligned}
-\int_\Omega \boldsymbol{\Delta}_{\mathcal{E}} \boldsymbol{u} \cdot \Pi_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} &= \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \sum_{K \in \mathcal{M}} \int_K (\boldsymbol{\nabla} \boldsymbol{u} . \boldsymbol{\nabla} \zeta_\sigma) \cdot \boldsymbol{v}_\sigma \,\mathrm{d}\boldsymbol{x} \\
&= \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \sum_{K \in \mathcal{M}} \int_K \boldsymbol{\nabla} \boldsymbol{u} : (\boldsymbol{v}_\sigma \otimes \boldsymbol{\nabla} \zeta_\sigma) \,\mathrm{d}\boldsymbol{x} \\
&= \sum_{K \in \mathcal{M}} \int_K \boldsymbol{\nabla} \boldsymbol{u} : \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} (\boldsymbol{v}_\sigma \otimes \boldsymbol{\nabla} \zeta_\sigma) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Since $\boldsymbol{v}(\boldsymbol{x}) = \sum_{\sigma \in \mathcal{E}} \boldsymbol{v}_\sigma \zeta_\sigma(\boldsymbol{x})$ and $\boldsymbol{v} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$, it is easy to check that $\boldsymbol{\nabla} \boldsymbol{v} = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} (\boldsymbol{v}_\sigma \otimes \boldsymbol{\nabla} \zeta_\sigma)$ wherever $\boldsymbol{v}$ is smooth. Hence:

$$-\int_\Omega \boldsymbol{\Delta}_{\mathcal{E}} \boldsymbol{u} \cdot \Pi_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \sum_{K \in \mathcal{M}} \int_K \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{u} : \boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}.$$

□

Thanks to these formulas we easily show the next lemma which corresponds to a discrete counterpart of the weak formulation of the momentum equation.

Lemma 4.10 (Weak formulation of the momentum balance - first form)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. A pair $(\rho, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M}}(\Omega)$ satisfies the discrete momentum balance (3.4b) if and only if:*

$$\int_\Omega \mathbf{div}_{\mathcal{E}}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \Pi_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{u} : \boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + (\mu + \lambda) \int_\Omega \mathrm{div}_{\mathcal{M}}\, \boldsymbol{u}\, \mathrm{div}_{\mathcal{M}}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}$$
$$- a \int_\Omega \rho^\gamma \,\mathrm{div}_{\mathcal{M}}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - h_{\mathcal{M}}^{\xi_3} \int_\Omega \rho^\Gamma \,\mathrm{div}_{\mathcal{M}}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \Pi_{\mathcal{E}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{v} \in \mathbf{H}_{\mathcal{M},0}(\Omega). \tag{4.17}$$

Proof. Multiplying (3.4b) by $|D_\sigma| \mathcal{X}_{D_\sigma}$, summing over $\sigma \in \mathcal{E}_{\mathrm{int}}$ and using the discrete Stokes identities (4.14), (4.15) and (4.16), we obtain (4.17). Conversely, multiplying (4.17) by $|D_\sigma|^{-1} \mathcal{X}_{D_\sigma}$ for all $\sigma \in \mathcal{E}_{\mathrm{int}}$ yields (3.4b). □

From this point, we assume that Γ and (ξ_1, ξ_2, ξ_3) satisfy the conditions (3.19)-(3.20)-(3.21).

Proposition 4.11 (Estimate on the discrete velocity)**.** *Let $(\rho, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M},0}(\Omega)$ be a solution of the numerical scheme (3.4). Then, we have $\rho > 0$ a.e. in Ω (i.e. $\rho_K > 0$, $\forall K \in \mathcal{M}$), and if $h_{\mathcal{M}} \le h_0$ (with h_0 depending on $\mu, \rho^\star, \Omega, \theta_0$), there exists $C_1 = C_1(\boldsymbol{f}, \mu, \rho^\star, \Omega, \xi_1, \theta_0)$ such that:*

$$\|\boldsymbol{u}\|_{1,2,\mathcal{M}}^2 \le C_1. \tag{4.18}$$

Proof. We take $\boldsymbol{u}$ as a test function in (4.17):

$$\int_\Omega \mathbf{div}_{\mathcal{E}}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \Pi_{\mathcal{E}} \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} + \mu \|\boldsymbol{u}\|_{1,2,\mathcal{M}}^2 + (\mu + \lambda) \|\mathrm{div}_{\mathcal{M}}\, \boldsymbol{u}\|_{\mathrm{L}^2(\Omega)}^2$$
$$- a \int_\Omega \rho^\gamma \,\mathrm{div}_{\mathcal{M}}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} - h_{\mathcal{M}}^{\xi_3} \int_\Omega \rho^\Gamma \,\mathrm{div}_{\mathcal{M}}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \Pi_{\mathcal{E}} \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}.$$

By remark 4.2 applied with $\beta = \gamma > 1$ and $\beta = \Gamma > 1$, the last two terms in the left hand side of this equality are seen to be non-negative. We thus obtain:

$$\int_\Omega \mathbf{div}_{\mathcal{E}}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \Pi_{\mathcal{E}} \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} + \mu \|\boldsymbol{u}\|_{1,2,\mathcal{M}}^2 \le \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \|\Pi_{\mathcal{E}} \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}. \tag{4.19}$$

Recalling that in the definition of the convection term, $\boldsymbol{u}_\epsilon = \frac{1}{2}(\boldsymbol{u}_\sigma + \boldsymbol{u}_{\sigma'})$ for $\epsilon = D_\sigma | D_{\sigma'}$, we get:

$$\int_\Omega \mathbf{div}_{\mathcal{E}}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \Pi_{\mathcal{E}} \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} = \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \Big(\sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\rho, \boldsymbol{u}) \Big) |\boldsymbol{u}_\sigma|^2$$
$$+ \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \Big(\sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \\ \epsilon = D_\sigma | D_{\sigma'}}} F_{\sigma,\epsilon}(\rho, \boldsymbol{u})\ \boldsymbol{u}_\sigma \cdot \boldsymbol{u}_{\sigma'} \Big),$$

and the last term in the right hand side vanishes thanks to the conservativity of the dual fluxes (assumption (H2)). Using the mass conservation equation satisfied on the dual mesh (3.15) in the first term, we get (denoting $\tilde{\rho}$ the piecewise constant scalar function which is equal to ρ_{D_σ} on every dual cell D_σ, and which satisfies $\tilde{\rho} > 0$ (because $\rho > 0$) and $\int_\Omega \tilde{\rho}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\,\mathrm{d}\boldsymbol{x} = |\Omega|\rho^\star$):

$$\Big| \int_\Omega \mathbf{div}_{\mathcal{E}}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \Pi_{\mathcal{E}} \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \Big| = \frac{1}{2}\, h_{\mathcal{M}}^{\xi_1} \Big| \int_\Omega (\tilde{\rho} - \rho^\star)\, |\Pi_{\mathcal{E}} \boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} \Big| \le h_{\mathcal{M}}^{\xi_1}\, |\Omega|\, \rho^\star\, \|\Pi_{\mathcal{E}} \boldsymbol{u}\|^2_{\mathbf{L}^\infty(\Omega)}.$$

Injecting in (4.19) yields:

$$\mu \|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \le \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)}\, \|\Pi_{\mathcal{E}} \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} + h_{\mathcal{M}}^{\xi_1} |\Omega|\, \rho^\star \|\Pi_{\mathcal{E}} \boldsymbol{u}\|^2_{\mathbf{L}^\infty(\Omega)}.$$

Thanks to the continuity of operator $\Pi_{\mathcal{E}}$: $\|\Pi_{\mathcal{E}} \boldsymbol{u}\|_{\mathbf{L}^q(\Omega)} \le C \|\boldsymbol{u}\|_{\mathbf{L}^q(\Omega)}$, to the inverse inequality $\|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)} \le h_{\mathcal{M}}^{-\frac{1}{2}} \|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)}$, and to the discrete Sobolev inequality $\|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} \le C \|\boldsymbol{u}\|_{1,2,\mathcal{M}}$ (with C only depending on Ω and θ_0), we obtain:

$$\mu \|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \le C \Big(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)}\, \|\boldsymbol{u}\|_{1,2,\mathcal{M}} + h_{\mathcal{M}}^{\xi_1 - 1} |\Omega|\, \rho^\star \|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \Big).$$

Applying Young's inequality, we get that for all $\kappa > 0$:

$$(\mu - C\kappa - C h_{\mathcal{M}}^{\xi_1 - 1} |\Omega|\, \rho^\star)\, \|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \le \frac{C}{4\kappa}\, \|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)}.$$

Since $\xi_1 > 1$, taking $h_{\mathcal{M}}$ and κ small enough yields:

$$\|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \le C_1,$$

where C_1 only depends on $\boldsymbol{f}$, μ, $\rho^\star$, Ω, ξ_1 and θ_0.

□

Remark 4.5. *The control on the discrete velocity is also independent of the space dimension. It is valid for $d = 2$ with the less restrictive assumption $\xi_1 > 0$.*

4.4 Estimates on the discrete density

We derive an estimate on the discrete pressure following the same lines as in the continuous setting. One remarkable property of the staggered discretization is the existence of a discrete analogue to the Bogovskii operator, which is also equivalent to an L^q *inf-sup* property satisfied by discrete functions. We refer to the appendix, Section C.3, for the proof (see also [21] for a proof which concerns the MAC scheme).

Lemma 4.12 (L^q Discrete inf-sup property)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. Then, there exists a linear operator*

$$\mathcal{B}_{\mathcal{M}} : \mathrm{L}_{\mathcal{M},0}(\Omega) \longrightarrow \mathbf{H}_{\mathcal{M},0}(\Omega)$$

depending only on Ω and on the discretization such that the following properties hold:

(i) For all $p \in \mathrm{L}_{\mathcal{M},0}(\Omega)$,

$$\int_\Omega r\, \mathrm{div}_{\mathcal{M}}(\mathcal{B}_{\mathcal{M}} p)\,\mathrm{d}\boldsymbol{x} = \int_\Omega r\, p\,\mathrm{d}\boldsymbol{x}, \qquad \forall r \in \mathrm{L}_{\mathcal{M}}(\Omega).$$

(ii) For all $q \in (1, +\infty)$, there exists $C = C(q, d, \Omega, \theta_0)$, such that

$$\|\mathcal{B}_{\mathcal{M}} p\|_{1,q,\mathcal{M}} \leq C \|p\|_{\mathrm{L}^q(\Omega)}.$$

Before deriving the control of the discrete pressure, we first present a second form of the weak formulation of the momentum equation which will be more convenient to handle.

Lemma 4.13 (Weak formulation of the momentum balance - second form)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω in the sense of Definition 3.1. The discrete weak formulation of the momentum balance (4.17) can be written in the following form:*

$$\begin{aligned} -\int_\Omega (\mathcal{P}_{\mathcal{E}}\rho)(\Pi_{\mathcal{E}}\boldsymbol{u}) \otimes (\Pi_{\mathcal{E}}\boldsymbol{u}) : \boldsymbol{\nabla}_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ + \mu \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}}\boldsymbol{u} : \boldsymbol{\nabla}_{\mathcal{M}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + (\mu+\lambda)\int_\Omega \mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ - a\int_\Omega \rho^\gamma\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - h_{\mathcal{M}}^{\xi_3}\int_\Omega \rho^\Gamma\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + R_{\mathrm{conv}}(\rho,\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\Pi_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.20}$$

where

$$R_{\mathrm{conv}}(\rho,\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mathbf{div}_{\mathcal{E}}(\rho\boldsymbol{u}\otimes\boldsymbol{u})\cdot\Pi_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega (\mathcal{P}_{\mathcal{E}}\rho)(\Pi_{\mathcal{E}}\boldsymbol{u})\otimes(\Pi_{\mathcal{E}}\boldsymbol{u}) : \boldsymbol{\nabla}_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}.$$

Assuming that $\gamma \in (\frac{3}{2}, 3]$, $\theta_{\mathcal{M}} \leq \theta_0$ and $h_{\mathcal{M}} \leq 1$, the remainder term satisfies the following estimate for some constant $C = C(\Omega, \gamma, \Gamma, \theta_0)$:

$$\begin{aligned} \big|R_{\mathrm{conv}}(\rho,\boldsymbol{u},\boldsymbol{v})\big| \leq C\, h_{\mathcal{M}}^{\frac{1}{2}-\frac{1}{\Gamma}(\frac{3}{1+\eta}+\xi_3)} \|h_{\mathcal{M}}^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\Gamma}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}}^2\, \|\boldsymbol{v}\|_{1,2,\mathcal{M}} \\ + C\, h_{\mathcal{M}}^{\xi_2-\frac{1}{\eta}-\frac{1}{\eta\Gamma}(\frac{3}{1+\eta}+\xi_3)} \|h_{\mathcal{M}}^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\eta\Gamma}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}}\, \|\boldsymbol{v}\|_{1,2,\mathcal{M}} \end{aligned} \tag{4.21}$$

Proof. This result is proved in Appendix A.2 (Lemma A.2). ☐

Remark 4.6. *Note that in the previous inequality (4.21), under the conditions (3.20)-(3.21) (since $\frac{\eta}{1+\eta} \leq \frac{1}{2}$), we guarantee that:*

$$\begin{aligned} \frac{1}{2} - \frac{1}{\Gamma}\left(\frac{3}{1+\eta} + \xi_3\right) &> \frac{\eta}{1+\eta} - \frac{5}{4\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right), \\ \xi_2 - \frac{1}{\eta} - \frac{1}{\eta\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right) &> \xi_2 - \frac{1}{\eta} - \frac{5}{4\eta\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right), \end{aligned}$$

so that the exponents of $h_{\mathcal{M}}$ appearing in (4.21) are positive under the assumptions (3.20)-(3.21).

We may now prove the following result which states mesh independent estimates satisfied by the discrete density when $(\rho, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M},0}(\Omega)$ is a solution of the numerical scheme (3.4).

Proposition 4.14. *Let $(\rho, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M},0}(\Omega)$ be a solution of the numerical scheme (3.4). Then, we have the following estimates:*

- *There exists $C_2 = C_2(\boldsymbol{f}, \mu, \lambda, \rho^\star, \Omega, \gamma, \Gamma, \xi_1, \xi_2, \xi_3, \theta_0)$ such that:*

$$\|\rho\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)} + \|h_{\mathcal{M}}^{\xi_3} \rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)} \leq C_2. \tag{4.22}$$

- *There exists $C_3 = C_3(\boldsymbol{f}, \mu, \lambda, \rho^\star, \Omega, \gamma, \Gamma, \xi_1, \xi_2, \xi_3, \theta_0)$ such that:*

$$\sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} |\sigma| \, (\rho_L - \rho_K)^2 \, |\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}| + h_{\mathcal{M}}^{\xi_2} \, |\rho|_{\frac{1+\eta}{\eta}, \mathcal{M}}^{\frac{1+\eta}{\eta}} \leq C_3 \, h_{\mathcal{M}}^{-\frac{5}{4\Gamma}\left(\frac{3}{1+\eta} + \xi_3\right)}. \tag{4.23}$$

Remark 4.7. *Note that from (4.22) we can easily deduce by interpolation between Lebesgue spaces that for all p with $1 \leq p < 1 + \eta$, there exists $r \in [0, 1)$ depending on p and γ (with $r = 0$ if $p = 1$) such that:*

$$\|h_{\mathcal{M}}^{\xi_3} \rho^\Gamma\|_{\mathrm{L}^p(\Omega)} \leq h_{\mathcal{M}}^{\xi_3} \|\rho^\Gamma\|_{\mathrm{L}^1(\Omega)}^{1-r} \|\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{r} \leq h_{\mathcal{M}}^{\xi_3(1-r)} \|\rho^\Gamma\|_{\mathrm{L}^1(\Omega)}^{1-r} \|h_{\mathcal{M}}^{\xi_3} \rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{r} \leq C_4 \, h_{\mathcal{M}}^{\xi_3(1-r)}, \tag{4.24}$$

with $C_4 = C_4(\boldsymbol{f}, \mu, \lambda, \rho^\star, \Omega, \gamma, \Gamma, \xi_1, \xi_2, \xi_3, \theta_0, p)$.

Remark 4.8. *Estimate (4.23) combines a so-called "weak BV estimate" and an estimate on the discrete H^1-semi norm of the density. Actually, as shown in the proof thereafter, if $\gamma \geq \frac{7}{3}$ we have:*

$$\sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} |\sigma| \, (\rho_L - \rho_K)^2 \, |\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}| + h_{\mathcal{M}}^{\xi_2} \, |\rho|_{\frac{1+\eta}{\eta}, \mathcal{M}}^{\frac{1+\eta}{\eta}} \leq \; C_4.$$

In the rest of the paper, we will not distinguish the cases $\gamma \geq \frac{7}{3}$ and $\gamma < \frac{7}{3}$, and we will assume that we are in the worst case, i.e. that we have (4.23).

Proof. Let us set $P(\rho) = a\rho^\gamma + h_{\mathcal{M}}^{\xi_3} \rho^\Gamma$. Similarly to the continuous case, we apply Lemma 4.12 to $P(\rho)^\eta - < P(\rho)^\eta >$ and we define $\boldsymbol{v} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$ by $\boldsymbol{v} = \mathcal{B}_{\mathcal{M}}(P(\rho)^\eta - < P(\rho)^\eta >)$. There exists $C = C(\Omega, \gamma, \theta_0)$ such that

$$\begin{aligned}
\|\boldsymbol{v}\|_{1,2,\mathcal{M}} &\leq C \|P(\rho)^\eta - < P(\rho)^\eta >\|_{\mathrm{L}^2(\Omega)} \\
&\leq C\Big(\|P(\rho)^{2\eta}\|_{\mathrm{L}^1(\Omega)}^{\frac{1}{2}} + \frac{1}{|\Omega|^{\frac{1}{2}}} \|P(\rho)^\eta\|_{\mathrm{L}^1(\Omega)} \Big) \\
&\leq C \|P(\rho)\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\eta}
\end{aligned}$$

where in the last step we have used Hölder's inequality in both terms (we recall that $\eta = \frac{2\gamma-3}{\gamma} \leq 1$ for $\gamma \leq 3$). With the same arguments, we have

$$\|\boldsymbol{v}\|_{1, \frac{1+\eta}{\eta}, \mathcal{M}} \leq C \|P(\rho)\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\eta}.$$

Taking $\boldsymbol{v}$ as a test function in (4.20), we obtain:

$$
\begin{aligned}
\int_\Omega (P(\rho))^{1+\eta}\,\mathrm{d}\boldsymbol{x} &= < P(\rho)^\eta > \Big(\int_\Omega P(\rho)\,\mathrm{d}\boldsymbol{x}\Big) \\
&\quad - \int_\Omega (\mathcal{P}_\mathcal{E}\rho)(\Pi_\mathcal{E}\boldsymbol{u}) \otimes (\Pi_\mathcal{E}\boldsymbol{u}) : \boldsymbol{\nabla}_\mathcal{E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \boldsymbol{\nabla}_\mathcal{M}\boldsymbol{u} : \boldsymbol{\nabla}_\mathcal{M}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\
&\quad + (\mu+\lambda)\int_\Omega \mathrm{div}_\mathcal{M}\,\boldsymbol{u}\,\mathrm{div}_\mathcal{M}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}\cdot\Pi_\mathcal{E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + R_{\mathrm{conv}}(\rho,\boldsymbol{u},\boldsymbol{v}) \\
&= T_1 + \cdots + T_6.
\end{aligned} \tag{4.25}
$$

We estimate the T_i as follows. First for T_1, following the same calculation as before we have $C = C(a,\Omega,\gamma,\theta_0)$ such that:

$$
\begin{aligned}
|T_1| &= \Big(\int_\Omega P(\rho)\,\mathrm{d}\boldsymbol{x}\Big) < P(\rho)^\eta > \\
&\le C\Big(\int_\Omega P(\rho)\,\mathrm{d}\boldsymbol{x}\Big)\|P(\rho)\|^\eta_{\mathrm{L}^{1+\eta}(\Omega)}.
\end{aligned}
$$

The integral is controlled as follows

$$
\begin{aligned}
\int_\Omega P(\rho)\,\mathrm{d}\boldsymbol{x} &\le a\,\|\rho\|^\gamma_{\mathrm{L}^\gamma(\Omega)} + h_n^{\xi_3}\|\rho\|^\Gamma_{\mathrm{L}^\Gamma(\Omega)} \\
&\le C\Big(\|\rho\|^{\gamma(1-r_1)}_{\mathrm{L}^1(\Omega)}\|\rho\|^{\gamma r_1}_{\mathrm{L}^{\gamma(1+\eta)}(\Omega)} + h_n^{\xi_3}\|\rho\|^{\Gamma(1-r_2)}_{\mathrm{L}^1(\Omega)}\|\rho\|^{\Gamma r_2}_{\mathrm{L}^{\Gamma(1+\eta)}(\Omega)}\Big)
\end{aligned}
$$

where we have used interpolation inequalities with $r_1,\ r_2 \in (0,1)$ such that

$$
\frac{1}{\gamma} = (1-r_1) + \frac{r_1}{\gamma(1+\eta)}, \qquad \frac{1}{\Gamma} = (1-r_2) + \frac{r_2}{\Gamma(1+\eta)}.
$$

Hence, by a Young inequality, we have $C = C(\Omega,\gamma,\Gamma,\xi_3,\theta_0)$ such that:

$$
|T_1| \le C + \frac{1}{5}\|P(\rho)\|^{1+\eta}_{\mathrm{L}^{1+\eta}(\Omega)}.
$$

The second term is controlled as follows, with $C = C(\Omega,\gamma,\theta_0)$:

$$
\begin{aligned}
|T_2| &= \Big|\int_\Omega (\mathcal{P}_\mathcal{E}\rho)(\Pi_\mathcal{E}\boldsymbol{u}) \otimes (\Pi_\mathcal{E}\boldsymbol{u}) : \boldsymbol{\nabla}_\mathcal{E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\Big| \\
&\le C\,\|\mathcal{P}_\mathcal{E}\rho\|_{\mathrm{L}^{\gamma(1+\eta)}(\Omega)}\,\|\Pi_\mathcal{E}\boldsymbol{u}\|^2_{\mathbf{L}^6(\Omega)}\,\|\boldsymbol{v}\|_{1,\frac{1+\eta}{\eta},\mathcal{M}} \\
&\le C\,\|P(\rho)\|^{1/\gamma}_{\mathrm{L}^{1+\eta}(\Omega)}\,\|\boldsymbol{u}\|^2_{\mathbf{L}^6(\Omega)}\,\|P(\rho)\|^\eta_{\mathrm{L}^{1+\eta}(\Omega)} \\
&\le C + \frac{1}{5}\|P(\rho)\|^{1+\eta}_{\mathrm{L}^{1+\eta}(\Omega)}.
\end{aligned}
$$

Next, observing that $\|\mathrm{div}_\mathcal{M}\,\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \le \sqrt{3}\,\|\boldsymbol{v}\|_{1,2,\mathcal{M}}$ for all $\boldsymbol{v} \in \mathbf{H}_\mathcal{M}(\Omega)$, we have $C = C(\Omega,\gamma,\theta_0)$ such that:

$$
|T_3| + |T_4| \le (\mu + 3(\mu+\lambda))\,\|\boldsymbol{u}\|_{1,2,\mathcal{M}}\,\|\boldsymbol{v}\|_{1,2,\mathcal{M}} \le C\,C_1\,(\mu+3(\mu+\lambda))\,\|P(\rho)\|^\eta_{\mathrm{L}^{1+\eta}(\Omega)}.
$$

By the discrete Poincaré inequality, the term T_5 satisfies with $C = C(\Omega, \gamma, \theta_0)$:

$$\begin{aligned}|T_5| &\le \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \|\Pi_{\mathcal{E}} \boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \\ &\le C \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \\ &\le C \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{v}\|_{1,2,\mathcal{M}} \\ &\le C \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \|P(\rho)\|^{\eta}_{\mathrm{L}^{1+\eta}(\Omega)}.\end{aligned}$$

Hence we get:

$$|T_3| + |T_4| + |T_5| \le C + \frac{1}{5}\|P(\rho)\|^{1+\eta}_{\mathrm{L}^{1+\eta}(\Omega)}.$$

The last term T_6 is the remainder term $R_{\mathrm{conv}}(\rho, \boldsymbol{u}, \boldsymbol{v})$ in the weak formulation of the momentum balance (4.20). We have thanks to (4.21), with $C = (\Omega, \gamma, \Gamma, \theta_0)$:

$$\begin{aligned}|T_6| = |R_{\mathrm{conv}}(\rho, \boldsymbol{u}, \boldsymbol{v})| \le C\, h_{\mathcal{M}}^{\frac{1}{2}-\frac{1}{\Gamma}(\frac{3}{1+\eta}+\xi_3)} \|h_{\mathcal{M}}^{\xi_3}\rho^{\Gamma}\|^{\frac{1}{\Gamma}}_{\mathrm{L}^{1+\eta}(\Omega)} \|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \|\boldsymbol{v}\|_{1,2,\mathcal{M}} \\ + C\, h_{\mathcal{M}}^{\xi_2-\frac{1}{\eta}-\frac{1}{\eta\Gamma}(\frac{3}{1+\eta}+\xi_3)} \|h_{\mathcal{M}}^{\xi_3}\rho^{\Gamma}\|^{\frac{1}{\eta\Gamma}}_{\mathrm{L}^{1+\eta}(\Omega)} \|\boldsymbol{u}\|_{1,2,\mathcal{M}} \|\boldsymbol{v}\|_{1,2,\mathcal{M}}.\end{aligned}$$

As a consequence of Remark 4.6, there exists $\nu > 0$ such that

$$\begin{aligned}|T_6| &\le Ch^{\nu}_{\mathcal{M}} \|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \|h_{\mathcal{M}}^{\xi_3}\rho^{\Gamma}\|^{\frac{1}{\Gamma}}_{\mathrm{L}^{1+\eta}(\Omega)} \|\boldsymbol{v}\|_{1,2,\mathcal{M}} + Ch^{\nu}_{\mathcal{M}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}} \|h_{\mathcal{M}}^{\xi_3}\rho^{\Gamma}\|^{\frac{1}{\eta\Gamma}}_{\mathrm{L}^{1+\eta}(\Omega)} \|\boldsymbol{v}\|_{1,2,\mathcal{M}} \\ &\le Ch^{\nu}_{\mathcal{M}} \|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \|h_{\mathcal{M}}^{\xi_3}\rho^{\Gamma}\|^{\frac{1}{\Gamma}}_{\mathrm{L}^{1+\eta}(\Omega)} \|P(\rho)\|^{\eta}_{\mathrm{L}^{1+\eta}(\Omega)} + Ch^{\nu}_{\mathcal{M}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}} \|h_{\mathcal{M}}^{\xi_3}\rho^{\Gamma}\|^{\frac{1}{\eta\Gamma}}_{\mathrm{L}^{1+\eta}(\Omega)} \|P(\rho)\|^{\eta}_{\mathrm{L}^{1+\eta}(\Omega)} \\ &\le Ch^{\nu}_{\mathcal{M}} \|\boldsymbol{u}\|^2_{1,2,\mathcal{M}} \|P(\rho)\|^{\eta+\frac{1}{\Gamma}}_{\mathrm{L}^{1+\eta}(\Omega)} + Ch^{\nu}_{\mathcal{M}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}} \|P(\rho)\|^{\eta+\frac{1}{\eta\Gamma}}_{\mathrm{L}^{1+\eta}(\Omega)} \\ &\le C + \frac{1}{5}\|P(\rho)\|^{1+\eta}_{\mathrm{L}^{1+\eta}(\Omega)},\end{aligned}$$

since $\frac{1}{\Gamma}$ and $\frac{1}{\eta\Gamma}$ are both less than 1 (consequence of (3.20)). Gathering the bounds on T_1,..,T_6 and coming back to (4.25) we get:

$$\int_{\Omega} (P(\rho))^{1+\eta}\,\mathrm{d}\boldsymbol{x} \le C + \frac{4}{5}\int_{\Omega} (P(\rho))^{1+\eta}\,\mathrm{d}\boldsymbol{x}.$$

This achieves the proof of (4.22).

It remains to prove (4.23). Taking $\beta = 2$ in the discrete renormalization identity (4.13), we get:

$$\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}} \\ \sigma=K|L}} |\sigma|\,(\rho_L-\rho_K)^2\,|\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}| + h_{\mathcal{M}}^{\xi_2}|\rho|^{\frac{1+\eta}{\eta}}_{\frac{1+\eta}{\eta},\mathcal{M}} \le -\int_{\Omega} \rho^2\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}$$

with, if $\gamma \ge \frac{7}{3}$ (and thus $3(\gamma-1) \ge 4$):

$$\begin{aligned}\left|\int_{\Omega} \rho^2\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\right| &\le \|\rho^2\|_{\mathrm{L}^2(\Omega)} \|\mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} \\ &\le \|\rho\|^2_{\mathrm{L}^4(\Omega)} \|\mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}\|_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

and otherwise (*i.e.* for $\gamma \in (\frac{3}{2}, \frac{7}{3})$):

$$\begin{aligned}\left|\int_\Omega \rho^2 \operatorname{div}_{\mathcal{M}} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}\right| &\leq \left|\int_\Omega \rho^{\frac{5}{4}} \rho^{\frac{3}{4}} \operatorname{div}_{\mathcal{M}} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}\right| \\ &\leq \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{5}{4}} \|\rho\|_{\mathrm{L}^{\frac{3}{2}}(\Omega)}^{\frac{3}{4}} \|\operatorname{div}_{\mathcal{M}} \boldsymbol{u}\|_{\mathrm{L}^2(\Omega)} \\ &\leq C(\Omega, \gamma, \Gamma, \theta_0) h_{\mathcal{M}}^{-\frac{5}{4\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \|h_{\mathcal{M}}^{\xi_3} \rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{5}{4\Gamma}} \|\rho\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)}^{\frac{3}{4}} \|\operatorname{div}_{\mathcal{M}} \boldsymbol{u}\|_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

This achieves the proof of (4.23).

□

4.5 Existence of a solution to the numerical scheme

The existence of a solution to the scheme (3.4), which consists in an algebraic non-linear system, is obtained by a topological degree argument. Its proof is based on an abstract theorem stated in Appendix B.1, which relies on linking by a homotopy the problem at hand to a linear system.

Let $N = \operatorname{card}(\mathcal{M})$ and $M = d \operatorname{card}(\mathcal{E}_{\mathrm{int}})$; we identify $\mathrm{L}_{\mathcal{M}}(\Omega)$ with $\mathbb{R}^N$ and $\mathbf{H}_{\mathcal{M},0}(\Omega)$ with $\mathbb{R}^M$. Let $V = \mathbb{R}^N \times \mathbb{R}^M$. We consider the function $\mathcal{F}: V \times [0,1] \to V$ given by:

$$\mathcal{F}(\rho, \boldsymbol{u}, \delta) = \left|\begin{array}{ll} \delta \dfrac{1}{|K|} \displaystyle\sum_{\sigma \in \mathcal{E}(K)} \overline{F}_{K,\sigma}(\rho, \boldsymbol{u}) + h_{\mathcal{M}}^{\xi_1} (\rho_K - \rho^\star), & K \in \mathcal{M} \\ \delta \dfrac{1}{|D_\sigma|} \displaystyle\sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\rho, \boldsymbol{u}) \boldsymbol{u}_\epsilon + \delta\, a (\boldsymbol{\nabla}_{\mathcal{E}}(\rho^\gamma))_{|D_\sigma} + \delta\, h_{\mathcal{M}}^{\xi_3} (\boldsymbol{\nabla}_{\mathcal{E}}(\rho^\Gamma))_{|D_\sigma} & \\ \qquad -\mu (\boldsymbol{\Delta}_{\mathcal{E}} \boldsymbol{u})_{|D_\sigma} - (\mu+\lambda) \big((\boldsymbol{\nabla} \circ \operatorname{div})_{\mathcal{E}} \boldsymbol{u}\big)_{|D_\sigma} - (\widetilde{\Pi}_{\mathcal{E}} \boldsymbol{f})_{|D_\sigma}, & \sigma \in \mathcal{E}_{\mathrm{int}}. \end{array}\right. \tag{4.26}$$

Solving the problem $\mathcal{F}(\rho, \boldsymbol{u}, \delta) = 0$ is equivalent to solving the following system analogous to (3.4):

Solve for $\rho \in \mathrm{L}_{\mathcal{M}}(\Omega)$ *and* $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$*:*

$$\delta \operatorname{div}_{\mathcal{M}}(\rho \boldsymbol{u}) + h_{\mathcal{M}}^{\xi_1} (\rho - \rho^\star) - \delta\, h_{\mathcal{M}}^{\xi_2} \Delta_{\frac{1+\eta}{\eta}, \mathcal{M}}(\rho) = 0, \tag{4.27a}$$

$$\delta \operatorname{\mathbf{div}}_{\mathcal{E}}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \boldsymbol{\Delta}_{\mathcal{E}} \boldsymbol{u} - (\mu+\lambda)(\boldsymbol{\nabla} \circ \operatorname{div})_{\mathcal{E}} \boldsymbol{u} + \delta\, a \boldsymbol{\nabla}_{\mathcal{E}}(\rho^\gamma) + \delta\, h_{\mathcal{M}}^{\xi_3} \boldsymbol{\nabla}_{\mathcal{E}}(\rho^\Gamma) = \widetilde{\Pi}_{\mathcal{E}} \boldsymbol{f}. \tag{4.27b}$$

Note that system (3.4) corresponds to $\mathcal{F}(\rho, \boldsymbol{u}, 1) = 0$. An easy verification shows that any solution $(\rho, \boldsymbol{u})$ of the problem $\mathcal{F}(\rho, \boldsymbol{u}, \delta) = 0$ for δ in $[0,1]$, satisfies the same estimates as stated in Propositions 4.7 (positivity of ρ) and 4.11 (estimate on $\|\boldsymbol{u}\|_{1,2,\mathcal{M}}$) uniformly in δ. However, the positivity of the density is not sufficient to apply the topological degree theorem stated in Appendix B.1. We need to prove that there exists a positive lower bound on ρ, if $(\rho, \boldsymbol{u})$ is a solution of (4.27), which is uniform with respect to $\delta \in [0,1]$. For the lower bound, we use Proposition 4.7 and the fact that $\|\boldsymbol{u}\|_{1,2,\mathcal{M}} \leq C_1$ uniformly with respect to $\delta \in [0,1]$ which implies that the quantity $\max\limits_{K \in \mathcal{M}} \operatorname{div}(\boldsymbol{u})_K$ is also controlled uniformly with respect to $\delta \in [0,1]$ as follows: there exists $\hat{K} \in \mathcal{M}$ such that

$$\begin{aligned}\left|\max_{K \in \mathcal{M}} \operatorname{div}(\boldsymbol{u})_K\right| = |\operatorname{div}(\boldsymbol{u})_{\hat{K}}| = \frac{1}{|\hat{K}|} \left|\int_{\hat{K}} \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}\right| &\leq |\hat{K}|^{-\frac{1}{2}} \|\operatorname{div} \boldsymbol{u}\|_{\mathrm{L}^2(\hat{K})} \\ &\leq \sqrt{3} |\hat{K}|^{-\frac{1}{2}} \|\boldsymbol{\nabla} \boldsymbol{u}\|_{\mathbf{L}^2(\hat{K})^3} \leq \sqrt{3} |\hat{K}|^{-\frac{1}{2}} \|\boldsymbol{\nabla} \boldsymbol{u}\|_{\mathbf{L}^2(\Omega)^3} \leq \sqrt{3 \max_{K \in \mathcal{M}} |K|^{-1}}\, C_1.\end{aligned}$$

Thus, a positive lower bound on ρ, if $(\rho, \boldsymbol{u})$ is a solution of (4.27), which is uniform with respect to $\delta \in [0,1]$, is given by:

$$\bar{\rho}_{\min} = \frac{\rho^\star}{1 + h_{\mathcal{M}}^{-\xi_1} \sqrt{3 \max\limits_{K \in \mathcal{M}} |K|^{-1} C_1}}. \tag{4.28}$$

We also obtain a uniform upper bound on ρ by remarking that:

$$\|\rho\|_{\mathrm{L}^\infty(\Omega)} \leq \max_{K \in \mathcal{M}} |K|^{-1} \|\rho\|_{\mathrm{L}^1(\Omega)} = \max_{K \in \mathcal{M}} |K|^{-1} |\Omega| \, \rho^\star =: \bar{\rho}_{\max}.$$

We may now prove the following result.

Theorem 4.15 (Existence of a solution)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω in the sense of Definition 3.1. The non-linear system* (3.4) *admits at least one solution $(\rho, \boldsymbol{u})$ in $\mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M},0}(\Omega)$, and any possible solution satisfies the estimates of Propositions 4.7, 4.11 and 4.14.*

Proof. This proof makes use of Theorem B.1. Let $\mathcal{F}$ be the function defined in (4.26). The function $\mathcal{F}$ is continuous from $V \times [0,1]$ to V. It therefore defines a homotopy between the problem $\mathcal{F}(\rho, \boldsymbol{u}, 1) = 0$ and $\mathcal{F}(\rho, \boldsymbol{u}, 0) = 0$. The first hypothesis of Theorem B.1 is satisfied and defining

$$\mathcal{O} = \Big\{ (\rho, \boldsymbol{u}) \in V \text{ s.t. } \frac{\bar{\rho}_{\min}}{2} < \rho < 2\, \bar{\rho}_{\max}, \ \|\boldsymbol{u}\|_{1,2,\mathcal{M}} < 2C_1 \Big\},$$

the second hypothesis of Theorem B.1 is also satisfied. Therefore, in order to prove the existence of at least one solution to the scheme (3.4), it remains to show that the topological degree of $\mathcal{F}(\rho, \boldsymbol{u}, 0)$ with respect to 0_V and $\mathcal{O}$ is non-zero. The function $G : (\rho, \boldsymbol{u}) \mapsto \mathcal{F}(\rho, \boldsymbol{u}, 0)$ is clearly differentiable on $\mathcal{O}$, and its jacobian matrix is given by

$$\text{Jac } G(\rho, \boldsymbol{u}) = \left[\begin{array}{c|c} h_{\mathcal{M}}^{\xi_1} \, \mathrm{Id}_{\mathbb{R}^{N \times N}} & 0 \\ \hline 0 & A \end{array} \right],$$

where $A \in \mathbb{R}^{M \times M}$ is the mass matrix associated with the finite element discretization of the following elliptic problem:

Find $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$ such that :

$$-\mu \boldsymbol{\Delta} \boldsymbol{u} - (\mu + \lambda) \boldsymbol{\nabla}(\operatorname{div} \boldsymbol{u}) = \boldsymbol{f}, \quad \text{in } \Omega. \tag{4.29}$$

By the coercivity of the discrete diffusion operator, the matrix A is invertible and the discretization of this elliptic problem has one and only one solution. Hence, by the first block of equations in $\mathcal{F}(\rho, \boldsymbol{u}, 0) = 0$, there exists one and only one point of $\mathcal{O}$ such that $\mathcal{F}(\rho, \boldsymbol{u}, 0) = 0$. Since the Jacobian matrix at this point Jac $G(\rho, \boldsymbol{u})$ is invertible (since $\mathrm{Id}_{\mathbb{R}^{N \times N}}$ and A are invertible), this implies that the topological degree of $\mathcal{F}(\rho, \boldsymbol{u}, 0)$ with respect to $\mathcal{O}$ and 0 is non-zero. Therefore, by Theorem B.1, there exists at least one solution $(\rho, \boldsymbol{u})$ to the equation $\mathcal{F}(\rho, \boldsymbol{u}, 1) = 0$, *i.e.* to the scheme (3.4). □

5 Proof of the convergence result

We recall the hypotheses of Theorem 3.1. Let Ω be a polyhedral connected open subset of $\mathbb{R}^3$. We assume that $\gamma \in (\frac{3}{2}, 3]$, $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ and $\rho^\star > 0$. Regarding our discretization, denoting $\eta = \frac{2\gamma-3}{\gamma} \in (0,1]$, we assume that Γ and (ξ_1, ξ_2, ξ_3) are such that:

$$(i) \qquad \xi_1 > 1, \tag{5.1}$$

$$(ii) \qquad \frac{5}{4\Gamma}\left(\frac{3}{1+\eta} + \xi_3\right) < \frac{\eta}{1+\eta} \ \leq \frac{1}{2} \tag{5.2}$$

$$(iii) \qquad \frac{1}{\eta} + \frac{5}{4\eta\Gamma}\left(\frac{3}{1+\eta} + \xi_3\right) < \xi_2 < \frac{1+\eta}{\eta} - \frac{5}{4\Gamma}\left(\frac{3}{1+\eta} + \xi_3\right). \tag{5.3}$$

Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a regular sequence of staggered discretizations of Ω as defined in Definition 3.3. We denote h_n instead of $h_{\mathcal{M}_n}$ in order to ease the notations. Similar simplifications will be used thereafter.

Theorem 4.15 applies and without loss of generality (assuming h_n is small enough for all $n \in \mathbb{N}$), we can assume that for all $n \in \mathbb{N}$ there exists a solution $(\rho_n, \boldsymbol{u}_n) \in \mathrm{L}_{\mathcal{M}_n}(\Omega) \times \mathbf{H}_{\mathcal{M}_n,0}(\Omega)$ to the numerical scheme (3.4) with the discretization $\mathcal{D}_n$. In addition, the obtained density ρ_n is positive *a.e.* in Ω. Since $\theta_{\mathcal{M}_n} \leq \theta_0$ for all $n \in \mathbb{N}$, the sequence $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ satisfies the following estimates. There exist $C_0 > 0$, $p \in (1, 1+\eta)$ and $r \in (0,1)$ such that:

$$\begin{aligned} &\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n} + \|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)} + \|h_n^{\xi_3}\rho_n^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)} + h_n^{\xi_2 + \frac{5}{4\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} |\rho_n|^{\frac{1+\eta}{\eta}}_{\frac{1+\eta}{\eta},\mathcal{M}_n} \\ &+ h_n^{-\xi_3(1-r)}\|h_n^{\xi_3}\rho_n^\Gamma\|_{\mathrm{L}^p(\Omega)} + h_n^{\frac{5}{4\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}} \\ \sigma=K|L}} |\sigma|\,(\rho_L - \rho_K)^2\,|\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}| \leq C_0, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.4}$$

In order to ease the notations, the subscript n has been omitted in the above summation on the internal faces of $\mathcal{E}_n$.

Thanks to these estimates, there is a subsequence of $(\mathcal{D}_n)_{n\in\mathbb{N}}$, still denoted $(\mathcal{D}_n)_{n\in\mathbb{N}}$ such that $(\rho_n)_{n\in\mathbb{N}}$ weakly converges in $\mathrm{L}^{3(\gamma-1)}(\Omega)$ to some $\rho \in \mathrm{L}^{3(\gamma-1)}(\Omega)$, and $(\rho_n^\gamma)_{n\in\mathbb{N}}$ weakly converges in $\mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$ to some $\overline{\rho^\gamma} \in \mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$. The compactness of the sequence of velocities relies on the following theorem (proven in Appendix C, see also [20], [38]) which is a compactness result for the discrete H^1_0-norm similar to Rellich's theorem.

Theorem 5.1 (Discrete Rellich theorem)**.** *Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of staggered discretizations of Ω satisfying $\theta_{\mathcal{M}_n} \leq \theta_0$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, let $\boldsymbol{u}_n \in \mathbf{H}_{\mathcal{M}_n,0}(\Omega)$ and assume that there exists $C \in \mathbb{R}$ such that $\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n} \leq C$, $\forall\, n \in \mathbb{N}$. We suppose that $h_n \to 0$ as $n \to +\infty$. Then:*

1. *There exists a subsequence of $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, still denoted $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, which converges in $\mathbf{L}^2(\Omega)$ towards a function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$.*

2. *The limit function $\boldsymbol{u}$ belongs to $\mathbf{H}^1_0(\Omega)$ with $\|\boldsymbol{\nabla}\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)^3} \leq C$.*

3. The sequence $(\boldsymbol{\nabla}_{\mathcal{M}_n}\boldsymbol{u}_n)_{n\in\mathbb{N}}$ *weakly converges to* $\boldsymbol{\nabla}\boldsymbol{u}$ *in* $\mathbf{L}^2(\Omega)^3$.

Hence, upon extracting a new subsequence from $(\mathcal{D}_n)_{n\in\mathbb{N}}$, we may assume that there exists $\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)$ such that the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{u}$ in $\mathbf{L}^2(\Omega)$. By the discrete Sobolev inequality of Lemma 4.1, we can actually assume that $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{u}$ in $\mathbf{L}^q(\Omega)$ for all $q \in [1, 6)$ and weakly in $\mathbf{L}^6(\Omega)$.

Following the same steps as in the proof of the stability property (Theorem 2.2) in the continuous setting, we first pass to the limit $n \to +\infty$ in the mass and momentum equations in Sections 5.1 and 5.2 and then pass to the limit in the equation of state in Section 5.3, by proving the strong convergence of the density.

5.1 Passing to the limit in the mass conservation equation

Proposition 5.2. *Under the assumptions of Theorem 3.1, the limit pair* $(\rho, \boldsymbol{u}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}^1_0(\Omega)$ *of the sequence* $(\rho_n, \boldsymbol{u}_n)_{n\in\mathbb{N}}$ *satisfies the mass equation in the weak sense:*

$$-\int_\Omega \rho\, \boldsymbol{u} \cdot \boldsymbol{\nabla}\phi \,\mathrm{d}\boldsymbol{x} = 0, \qquad \forall \phi \in \mathcal{C}^\infty_c(\Omega). \tag{5.5}$$

Let us first state the following lemma which will be useful in the proof of Proposition (5.2).

Lemma 5.3. *Let* $\phi \in \mathcal{C}^\infty_c(\Omega)$*. For* $n \in \mathbb{N}$ *define* $\phi_n \in \mathrm{L}_{\mathcal{M}_n}(\Omega)$ *by* $\phi_{n|K} = \phi_K$ *the mean value of* ϕ *over* K*, for* $K \in \mathcal{M}_n$*. Denote* $\phi_\sigma = |\sigma|^{-1}\int_\sigma \phi(\boldsymbol{x})\,\mathrm{d}\sigma(\boldsymbol{x})$ *for all* $\sigma \in \mathcal{E}_n$ *and define a discrete gradient of* ϕ_n *by:*

$$\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n(\boldsymbol{x}) = \sum_{K\in\mathcal{M}_n} (\boldsymbol{\nabla}\phi)_K\, \mathcal{X}_K(\boldsymbol{x}), \quad \text{with} \quad (\boldsymbol{\nabla}\phi)_K = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, \phi_\sigma\, \boldsymbol{n}_{K,\sigma}.$$

Then for all q *in* $[1,\infty]$*, there exists* $C = C(\Omega, q, \phi, \theta_0)$ *such that:*

$$\|\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n - \boldsymbol{\nabla}\phi\|_{\mathbf{L}^q(\Omega)} \le C\, h_n. \tag{5.6}$$

Proof. Let $q \in [1, +\infty)$. We have $\|\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n - \boldsymbol{\nabla}\phi\|^q_{\mathbf{L}^q(\Omega)} = \sum_{K\in\mathcal{M}_n} \|\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n - \boldsymbol{\nabla}\phi\|^q_{\mathbf{L}^q(K)}$ with for $K \in \mathcal{M}_n$:

$$\begin{aligned}
\|\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n - \boldsymbol{\nabla}\phi\|^q_{\mathbf{L}^q(K)} &= \int_K \Big|\frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\phi_\sigma\, \boldsymbol{n}_{K,\sigma} - \boldsymbol{\nabla}\phi(\boldsymbol{x})\Big|^q \mathrm{d}\boldsymbol{x} \\
&= \int_K \Big|\frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \phi(\boldsymbol{y})\,\mathrm{d}\sigma(\boldsymbol{y})\, \boldsymbol{n}_{K,\sigma} - \boldsymbol{\nabla}\phi(\boldsymbol{x})\Big|^q \mathrm{d}\boldsymbol{x} \\
&= \int_K \Big|\frac{1}{|K|}\int_{\partial K} \phi(\boldsymbol{y})\,\mathrm{d}\sigma(\boldsymbol{y})\, \boldsymbol{n}_K - \boldsymbol{\nabla}\phi(\boldsymbol{x})\Big|^q \mathrm{d}\boldsymbol{x} \\
&= \int_K \Big|\frac{1}{|K|}\int_K (\boldsymbol{\nabla}\phi(\boldsymbol{y}) - \boldsymbol{\nabla}\phi(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{y}\Big|^q \mathrm{d}\boldsymbol{x} \\
&\le \int_K \Big(\frac{1}{|K|}\int_K |\boldsymbol{\nabla}\phi(\boldsymbol{y}) - \boldsymbol{\nabla}\phi(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{y}\Big)^q \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

By a Taylor expansion, we have for all $\boldsymbol{y}, \boldsymbol{x} \in K$, $|\boldsymbol{\nabla}\phi(\boldsymbol{y}) - \boldsymbol{\nabla}\phi(\boldsymbol{x})| \leq h_n\, |\phi|_{\mathrm{W}^{2,\infty}(\Omega)}$. Thus we have: $\|\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n - \boldsymbol{\nabla}\phi\|^q_{\mathbf{L}^q(K)} \leq h_n^q\, |\phi|^q_{\mathrm{W}^{2,\infty}(\Omega)}\, |K|$ which concludes the proof for $q \in [1, +\infty)$. The proof is similar for $q = +\infty$. □

We can now give the proof of Proposition (5.2).

Proof of Proposition (5.2). To prove this result we pass to the limit $n \to +\infty$ in the weak formulation of the discrete mass balance. Let $\phi \in \mathcal{C}_c^\infty(\Omega)$ and for $n \in \mathbb{N}$ define $\phi_n \in \mathrm{L}_{\mathcal{M}_n}(\Omega)$ by $\phi_{n|K} = \phi_K$ the mean value of ϕ over K, for $K \in \mathcal{M}_n$. Multiplying the discrete mass balance (3.4a) by $|K|\,\phi_K\,\mathcal{X}_K$, summing over $K \in \mathcal{M}_n$ and performing a discrete integration by parts (*i.e.* reordering the sum) yields:

$$-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\cdot\boldsymbol{\nabla}_{\mathcal{E}_n}\phi_n\ \mathrm{d}\boldsymbol{x} + R_1^n + R_2^n = 0, \tag{5.7}$$

with

$$R_1^n = h_n^{\xi_1}\sum_{K\in\mathcal{M}_n}|K|(\rho_K - \rho^\star)\,\phi_K,$$
$$R_2^n = h_n^{\xi_2}\sum_{K\in\mathcal{M}_n}\Big(\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\,\Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}}\,|\rho_K-\rho_L|^{\frac{1}{\eta}-1}\,(\rho_K-\rho_L)\Big)\,\phi_K,$$

where $\boldsymbol{\nabla}_\mathcal{E}$ is the discrete gradient defined in (3.17) and $\mathcal{P}_\mathcal{E}$ is defined in (4.8).

In order to prove Proposition 5.2, we want to pass to the limit in the first term of (5.7). It is possible to prove that $(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) \to \boldsymbol{u}$ strongly in $\mathbf{L}^2(\Omega)$. However, the discrete gradient $\boldsymbol{\nabla}_{\mathcal{E}_n}\phi_n$ in known to converge only weakly towards $\boldsymbol{\nabla}\phi$ because locally on a dual cell D_σ it is supported by only one direction, that of the normal vector $\boldsymbol{n}_{K,\sigma}$, and since, at this stage, ρ_n converges only weakly towards ρ (in $\mathrm{L}^{3(\gamma-1)}(\Omega)$), we cannot expect more than a weak convergence for $\mathcal{P}_{\mathcal{E}_n}\rho_n$. Thus, it is not possible to pass to the limit in the present form of (5.7). Instead, we use the discrete gradient $\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n$ introduced in Lemma 5.3, which is known to converge strongly towards $\boldsymbol{\nabla}\phi$.

We have:

$$\begin{aligned}
-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\cdot\boldsymbol{\nabla}_{\mathcal{E}_n}\phi_n\ \mathrm{d}\boldsymbol{x} &= -\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}}|\sigma|\,\rho_\sigma\,(\phi_L-\phi_K)\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}\\
&= -\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}}|\sigma|\,\tfrac{1}{2}(\rho_K+\rho_L)\,(\phi_L-\phi_K)\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma} + R_3^n,
\end{aligned} \tag{5.8}$$

with

$$R_3^n = \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}}|\sigma|\,\big(\tfrac{1}{2}(\rho_K+\rho_L)-\rho_\sigma\big)\,(\phi_L-\phi_K)\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}.$$

Reordering the sum in the first term of (5.8) we get:

$$
\begin{aligned}
-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\cdot\boldsymbol{\nabla}_{\mathcal{E}_n}\phi_n \;\mathrm{d}\boldsymbol{x}&\\
&= -\frac{1}{2}\sum_{K\in\mathcal{M}_n}\rho_K\Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\,(\phi_L-\phi_K)\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}\Big)+R_3^n,\\
&= -\frac{1}{2}\sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\cdot\Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\,(\phi_L-\phi_K)\,\boldsymbol{n}_{K,\sigma}\Big)+R_3^n+R_4^n,
\end{aligned}
\tag{5.9}
$$

where $\boldsymbol{u}_K$ is the mean value of $\boldsymbol{u}_n$ over K and

$$
R_4^n=\frac{1}{2}\sum_{K\in\mathcal{M}_n}\rho_K\Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\,(\phi_L-\phi_K)\,(\boldsymbol{u}_K-\boldsymbol{u}_\sigma)\cdot\boldsymbol{n}_{K,\sigma}\Big).
$$

Back to (5.9), we have:

$$
\begin{aligned}
-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\cdot\boldsymbol{\nabla}_{\mathcal{E}_n}\phi_n \;\mathrm{d}\boldsymbol{x}&\\
&= -\sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\cdot\Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\,\tfrac{1}{2}(\phi_L+\phi_K)\,\boldsymbol{n}_{K,\sigma}\Big)+R_3^n+R_4^n+R_5^n\\
&= -\sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\cdot\Big(\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\,\phi_\sigma\,\boldsymbol{n}_{K,\sigma}\Big)+R_3^n+R_4^n+R_5^n+R_6^n\\
&= -\int_\Omega \rho_n\,\boldsymbol{u}_n\cdot\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi\,\mathrm{d}\boldsymbol{x}+R_3^n+R_4^n+R_5^n+R_6^n.
\end{aligned}
\tag{5.10}
$$

where :

$$
\begin{aligned}
R_5^n&=\sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\,\phi_K\cdot\Big(\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\,\boldsymbol{n}_{K,\sigma}\Big),\\
R_6^n&=\sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\cdot\Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}}|\sigma|\,(\phi_\sigma-\tfrac{1}{2}(\phi_L+\phi_K))\,\boldsymbol{n}_{K,\sigma}\Big).
\end{aligned}
$$

Replacing (5.10) in (5.7) we get:

$$
-\int_\Omega \rho_n\,\boldsymbol{u}_n\cdot\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi\,\mathrm{d}\boldsymbol{x}+R_1^n+R_2^n+R_3^n+R_4^n+R_5^n+R_6^n=0.
\tag{5.11}
$$

Since $\boldsymbol{u}_n\to\boldsymbol{u}$ strongly in $\mathbf{L}^q(\Omega)$ for all $q\in[1,6)$, and $\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n\to\boldsymbol{\nabla}\phi$ (by (5.6)) in $\mathbf{L}^6(\Omega)^3$ as $n\to+\infty$, we have $\boldsymbol{u}_n\cdot\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n\to\boldsymbol{u}\cdot\boldsymbol{\nabla}\phi$ strongly in $\mathrm{L}^{3-\delta}(\Omega)$ for all $\delta\in(0,2]$. Furthermore, we have $\rho_n\rightharpoonup\rho$ weakly in $\mathrm{L}^{3(\gamma-1)}(\Omega)$ with $3(\gamma-1)>\frac{3}{2}$ (since $\gamma>\frac{3}{2}$), which yields:

$$
\lim_{n\to+\infty}\int_\Omega \rho_n\,\boldsymbol{u}_n\cdot\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\phi_n\;\mathrm{d}\boldsymbol{x}=\int_\Omega \rho\,\boldsymbol{u}\cdot\boldsymbol{\nabla}\phi\,\mathrm{d}\boldsymbol{x}.
$$

It remains to prove that $\sum_{i=1}^{6} R_i^n \to 0$ as $n \to +\infty$. In the following, in order to ease the notations, we denote $A_n \lesssim B_n$ when there is a constant C, independent of n, such that $A_n \le C\, B_n$. We easily prove that $R_1^n \to 0$ and $R_2^n \to 0$ as $n \to +\infty$. Indeed, one has:

$$|R_1^n| \le 2\, h_n^{\xi_1}\, |\Omega| \rho^\star\, \|\phi\|_{\mathrm{L}^\infty(\Omega)},$$

which proves that $R_1^n \to 0$ since $\xi_1 > 0$. For R_2^n, reordering the sum, we get:

$$\begin{aligned} |R_2^n| &\le h_n^{\xi_2} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}}\, |\phi_K-\phi_L| \\ &\lesssim \|\boldsymbol{\nabla}\phi\|_{\mathbf{L}^\infty(\Omega)}\, h_n^{\xi_2} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |D_\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}}. \end{aligned}$$

Applying Hölder's inequality (with coefficients $1+\eta$ and $(1+\eta)/\eta$) to the sum, we get:

$$\begin{aligned} |R_2^n| &\lesssim \|\boldsymbol{\nabla}\phi\|_{\mathbf{L}^\infty(\Omega)}\, |\Omega|^{\frac{\eta}{\eta+1}}\, h_n^{\xi_2} |\rho|_{\frac{1+\eta}{\eta}}^{\frac{1}{\eta}} \\ &\lesssim h_n^{\frac{\eta}{1+\eta}(\xi_2-\frac{5}{4\eta\Gamma}(\frac{3}{1+\eta}+\xi_3))} \Big(h_n^{\frac{\eta}{1+\eta}(\xi_2+\frac{5}{4\Gamma}(\frac{3}{1+\eta}+\xi_3))} |\rho|_{\frac{1+\eta}{\eta}}\Big)^{\frac{1}{\eta}}. \end{aligned}$$

Thanks to (5.4) and to assumption (5.3), we have $R_2^n \to 0$ as $n \to +\infty$. Let us now turn to R_3^n. Recalling the upwind definition of ρ_σ, we get:

$$|R_3^n| \le \frac{1}{2} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma|\, |\rho_K-\rho_L|\, |\phi_L-\phi_K|\, |\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}|.$$

Applying the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} |R_3^n| &\le \frac{1}{2} \Big(\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma|\, |\rho_K-\rho_L|^2\, |\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}|\Big)^{\frac{1}{2}} \Big(\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma|\, |\phi_L-\phi_K|^2\, |\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}|\Big)^{\frac{1}{2}} \\ &\lesssim h_n^{-\frac{5}{8\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \Big(\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma|\, |\phi_L-\phi_K|^2\, |\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}|\Big)^{\frac{1}{2}} \end{aligned}$$

by estimate (5.4). By Taylor's inequality applied to the smooth function ϕ and the regularity of the discretization, we have $|\phi_L-\phi_K|^2 \lesssim h_n\, |D_\sigma|/|\sigma|\, \|\boldsymbol{\nabla}\phi\|^2_{\mathbf{L}^\infty(\Omega)}$. Hence:

$$\begin{aligned} |R_3^n| &\lesssim h_n^{\frac{5}{8\Gamma}\left(\frac{4}{5}\Gamma-\frac{3}{1+\eta}-\xi_3\right)} \Big(\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |D_\sigma|\, |\boldsymbol{u}_\sigma|\Big)^{\frac{1}{2}} = h_n^{\frac{5}{8\Gamma}\left(\frac{4}{5}\Gamma-\frac{3}{1+\eta}-\xi_3\right)} \|\Pi_{\mathcal{E}_n}\boldsymbol{u}_n\|^{\frac{1}{2}}_{\mathbf{L}^1(\Omega)} \\ &\lesssim h_n^{\frac{5}{8\Gamma}\left(\frac{4}{5}\Gamma-\frac{3}{1+\eta}-\xi_3\right)} \|\boldsymbol{u}_n\|^{\frac{1}{2}}_{\mathbf{L}^1(\Omega)} \end{aligned}$$

since $\|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)}$, and thus $\|\boldsymbol{u}_n\|_{\mathbf{L}^1(\Omega)}$, is controlled by $\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n}$ which is bounded by C_0. Since (Γ, ξ_3) satisfy (5.2) we get $R_3^n \to 0$ as $n \to +\infty$. We now turn to R_4^n. By a Taylor inequality on the smooth function ϕ and the regularity of the discretization, we have: $|\phi_L - \phi_K| \lesssim h_n \|\boldsymbol{\nabla}\phi\|_{\mathbf{L}^\infty(\Omega)}$. Hence:

$$|R_4^n| \lesssim h_n \sum_{K\in\mathcal{M}_n} \rho_K \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, |\boldsymbol{u}_K - \boldsymbol{u}_\sigma|. \tag{5.12}$$

The vectors $\boldsymbol{u}_K$ and $\boldsymbol{u}_\sigma$ are the mean values of $\boldsymbol{u}$ over K and $\sigma \in \mathcal{E}(K)$ respectively. By the Cauchy-Schwarz inequality, we can prove that:

$$|\boldsymbol{u}_K - \boldsymbol{u}_\sigma|^2 \le \frac{1}{|\sigma|}\frac{1}{|K|}\int_\sigma\int_K |\boldsymbol{u}_n(\boldsymbol{y}) - \boldsymbol{u}_n(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\sigma(\boldsymbol{y}).$$

Since $\boldsymbol{u}_n$ is smooth over K we have for $\boldsymbol{x}$, $\boldsymbol{y} \in K$:

$$|\boldsymbol{u}_n(\boldsymbol{y}) - \boldsymbol{u}_n(\boldsymbol{x})|^2 \le |\boldsymbol{y}-\boldsymbol{x}|^2 \int_0^1 |\boldsymbol{\nabla}\boldsymbol{u}_n(t\boldsymbol{y} + (1-t)\boldsymbol{x})|^2 \, \mathrm{d}t.$$

Bounding $|\boldsymbol{y}-\boldsymbol{x}|$ by h_K we obtain, using Fubini's theorem that $|\boldsymbol{u}_K - \boldsymbol{u}_\sigma|^2 \le \frac{h_K^2}{|K|}\|\boldsymbol{\nabla}\boldsymbol{u}_n\|^2_{\mathbf{L}^2(K)^3}$. Injecting in (5.12), and invoking the regularity of the discretization we get:

$$\begin{aligned}
|R_4^n| &\lesssim h_n \sum_{K\in\mathcal{M}_n} |K|^{\frac{1}{2}} \rho_K \|\boldsymbol{\nabla}\boldsymbol{u}_n\|_{\mathbf{L}^2(K)^3} \\
&\lesssim h_n \|\rho_n\|_{\mathrm{L}^\infty(\Omega)}^{1-\frac{3(\gamma-1)}{2}} \sum_{K\in\mathcal{M}_n} |K|^{\frac{1}{2}} \rho_K^{\frac{3(\gamma-1)}{2}} \|\boldsymbol{\nabla}\boldsymbol{u}_n\|_{\mathbf{L}^2(K)^3} \\
&\lesssim h_n \|\rho_n\|_{\mathrm{L}^\infty(\Omega)}^{\frac{5-3\gamma}{2}} \|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)}^{\frac{3(\gamma-1)}{2}} \|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n}.
\end{aligned}$$

Thus, by the inverse inequality $\|\rho_n\|_{\mathrm{L}^\infty(\Omega)} \lesssim h_n^{-\frac{1}{\gamma-1}} \|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)}$ and since the sequence $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{3(\gamma-1)}(\Omega)$ and the sequence $(\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n})_{n\in\mathbb{N}}$ is bounded we get:

$$|R_4^n| \lesssim h_n^{1-\frac{1}{\gamma-1}\frac{5-3\gamma}{2}} = h_n^{\frac{5\gamma-7}{2(\gamma-1)}}.$$

Since, $\gamma > \frac{3}{2} > \frac{7}{5}$, we deduce that $R_4^n \to 0$ as $n \to +\infty$.
The fifth remainder term satisfies $R_5^n = 0$ since $\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, \boldsymbol{n}_{K,\sigma} = 0$ for all $K \in \mathcal{M}_n$. Let us conclude with the control of R_6^n. Denoting $\hat{\phi}_\sigma = \frac{1}{2}(\phi_L + \phi_K)$ for $\sigma = K|L$, we may write $R_6^n = R_{6,1}^n + R_{6,2}^n$ with:

$$\begin{aligned}
R_{6,1}^n &= \sum_{K\in\mathcal{M}_n} \rho_K \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, (\phi_\sigma - \hat{\phi}_\sigma)\, (\boldsymbol{u}_K - \boldsymbol{u}_\sigma)\cdot \boldsymbol{n}_{K,\sigma} \Big), \\
R_{6,2}^n &= \sum_{K\in\mathcal{M}_n} \rho_K \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, (\phi_\sigma - \hat{\phi}_\sigma)\, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} \Big).
\end{aligned}$$

The term $R_{6,1}^n$ can be controlled the same way as R_4^n and we obtain $R_{6,1}^n \to 0$ as $n \to +\infty$. Reordering the sum in $R_{6,2}^n$ we get:

$$R_{6,2}^n = \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}} \\ \sigma=K|L}} |\sigma|\, (\rho_K - \rho_L)\, (\phi_\sigma - \hat{\phi}_\sigma)\, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}.$$

Hence $R_{6,2}^n$ can be controlled the same way as R_3^n and we obtain $R_{6,2}^n \to 0$ as $n \to +\infty$ and this concludes the proof of (5.5).

□

5.2 Passing to the limit in the momentum equation

Proposition 5.4. *Under the assumptions of Theorem 3.1, the limit triple* $(\rho, \boldsymbol{u}, \overline{\rho^\gamma}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$ *of the sequence* $(\rho_n, \boldsymbol{u}_n, \rho_n^\gamma)_{n\in\mathbb{N}}$ *satisfies the momentum equation in the weak sense:*

$$-\int_\Omega \rho\, \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + (\mu + \lambda) \int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}$$
$$- a \int_\Omega \overline{\rho^\gamma} \operatorname{div} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{v} \in \mathcal{C}_c^\infty(\Omega)^3. \tag{5.13}$$

Moreover, we have the following energy inequality satisfied at the limit:

$$\mu \int_\Omega |\boldsymbol{\nabla} \boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} + (\lambda + \mu) \int_\Omega (\operatorname{div} \boldsymbol{u})^2 \,\mathrm{d}\boldsymbol{x} \le \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}. \tag{5.14}$$

Let us first state the following lemmas which will be useful in the proof of Proposition (5.4).

For $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ a staggered discretization of Ω, we define $I_\mathcal{M}$ the following Fortin operator associated with the Crouzeix-Raviart finite element:

$$I_\mathcal{M} : \begin{cases} \mathbf{W}^{1,q}(\Omega) & \longrightarrow \quad \mathbf{H}_\mathcal{M}(\Omega) \\ \quad \boldsymbol{v} & \longmapsto \quad I_\mathcal{M} \boldsymbol{v} = \displaystyle\sum_{\sigma \in \mathcal{E}} \boldsymbol{v}_\sigma\, \zeta_\sigma, \quad \text{ with } \boldsymbol{v}_\sigma = |\sigma|^{-1} \int_\sigma \boldsymbol{v} \,\mathrm{d}\sigma(\boldsymbol{x}) \text{ for } \sigma \in \mathcal{E}. \end{cases} \tag{5.15}$$

The following lemma states the main properties of operator $I_\mathcal{M}$. We refer to the appendix, Section C.2, for the proof. See also the appendix of [23].

Lemma 5.5 (Properties of the operator $I_\mathcal{M}$)**.** *Let* $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ *be a staggered discretization of* Ω *such that* $\theta_\mathcal{M} \le \theta_0$ *(where* $\theta_\mathcal{M}$ *is defined by (3.1)) for some positive constant* θ_0*. For any* $q \in [1, +\infty)$*, there exists* $C = C(\theta_0, q)$ *such that:*

(i) Stability:
$$\| I_\mathcal{M} \boldsymbol{u} \|_{1,q,\mathcal{M}} \le C\, |\boldsymbol{u}|_{\mathbf{W}^{1,q}(\Omega)}, \qquad \forall \boldsymbol{u} \in \mathbf{W}_0^{1,q}(\Omega).$$

(ii) Approximation: For all $K \in \mathcal{M}$*:*
$$\| \boldsymbol{u} - I_\mathcal{M} \boldsymbol{u} \|_{\mathbf{L}^q(K)} + h_K \, \| \boldsymbol{\nabla} (\boldsymbol{u} - I_\mathcal{M} \boldsymbol{u}) \|_{\mathbf{L}^q(K)^3}$$
$$\le C\, h_K^2\, |\boldsymbol{u}|_{\mathbf{W}^{2,q}(K)}, \qquad \forall \boldsymbol{u} \in \mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}_0^{1,q}(\Omega).$$

(iii) Preservation of the divergence:
$$\int_\Omega p \operatorname{div}_\mathcal{M}(I_\mathcal{M} \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega p \operatorname{div} \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}, \qquad \forall p \in \mathrm{L}_\mathcal{M}(\Omega),\ \boldsymbol{u} \in \mathbf{W}_0^{1,q}(\Omega).$$

Lemma 5.6. *Let $\boldsymbol{v} \in \mathcal{C}_c^\infty(\Omega)^3$. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a regular sequence of staggered discretizations as defined in Definition 3.3. For $n \in \mathbb{N}$ define $\boldsymbol{v}_n \in \mathbf{H}_{\mathcal{M}_n,0}(\Omega)$ by $\boldsymbol{v}_n = I_{\mathcal{M}_n}\boldsymbol{v}$. Then, for any $q \in [1, +\infty)$, there exists $C = C(\Omega, q, \boldsymbol{v}, \theta_0)$ such that:*

$$\|\boldsymbol{v}_n - \boldsymbol{v}\|_{\mathbf{L}^q(\Omega)} \leq C\, h_n^2, \tag{5.16}$$

$$\|\boldsymbol{\nabla}_{\mathcal{M}_n}\boldsymbol{v}_n - \boldsymbol{\nabla}\boldsymbol{v}\|_{\mathbf{L}^q(\Omega)^3} \leq C\, h_n, \tag{5.17}$$

$$\|\Pi_{\mathcal{E}_n}\boldsymbol{v}_n - \boldsymbol{v}\|_{\mathbf{L}^q(\Omega)} \leq C\, h_n. \tag{5.18}$$

In addition, denoting $\boldsymbol{v}_\sigma = |\sigma|^{-1}\int_\sigma \boldsymbol{v}\,\mathrm{d}\sigma(\boldsymbol{x})$ for all $\sigma \in \mathcal{E}_n$ we define a discrete gradient of $\boldsymbol{v}_n$ by:

$$\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\boldsymbol{v}_n(\boldsymbol{x}) = \sum_{K\in\mathcal{M}_n} (\boldsymbol{\nabla}\boldsymbol{v})_K\, \mathcal{X}_K(\boldsymbol{x}), \quad \textit{with} \quad (\boldsymbol{\nabla}\boldsymbol{v})_K = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, \boldsymbol{v}_\sigma \otimes \boldsymbol{n}_{K,\sigma}.$$

Then for all q in $[1, \infty]$, there exists $C = C(\Omega, q, \boldsymbol{v}, \theta_0)$ such that:

$$\|\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\boldsymbol{v}_n - \boldsymbol{\nabla}\boldsymbol{v}\|_{\mathbf{L}^q(\Omega)^3} \leq C\, h_n. \tag{5.19}$$

Proof. The estimates (5.16) and (5.17) are direct consequences of the approximation properties of the interpolation operator $I_{\mathcal{M}_n}$. The proof of (5.19) is similar to that of Lemma 5.3. Let us prove (5.18).

$$\begin{aligned}
\|\Pi_{\mathcal{E}_n}\boldsymbol{v}_n - \boldsymbol{v}_n\|_{\mathbf{L}^q(\Omega)}^q &= \sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)} \int_{D_{K,\sigma}} |\boldsymbol{v}_\sigma - \boldsymbol{v}_n(\boldsymbol{x})|^q\,\mathrm{d}\boldsymbol{x} \\
&= \sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)} \int_{D_{K,\sigma}} \Big|\boldsymbol{v}_\sigma - \sum_{\sigma'\in\mathcal{E}(K)} \boldsymbol{v}_{\sigma'}\,\zeta_{\sigma'}(\boldsymbol{x})\Big|^q\,\mathrm{d}\boldsymbol{x} \\
&= \sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)} \int_{D_{K,\sigma}} \Big|\sum_{\sigma'\in\mathcal{E}(K)} (\boldsymbol{v}_\sigma - \boldsymbol{v}_{\sigma'})\,\zeta_{\sigma'}(\boldsymbol{x})\Big|^q\,\mathrm{d}\boldsymbol{x} \\
&\lesssim h_n^q \sum_{K\in\mathcal{M}_n} h_K^{3-q} \sum_{\sigma,\sigma'\in\mathcal{E}(K)} |\boldsymbol{v}_\sigma - \boldsymbol{v}_{\sigma'}|^2.
\end{aligned}$$

Hence we have $\|\Pi_{\mathcal{E}_n}\boldsymbol{v}_n - \boldsymbol{v}_n\|_{\mathbf{L}^q(\Omega)} \lesssim h_n\,\|\boldsymbol{v}_n\|_{1,q,\mathcal{E}_n} \lesssim h_n\,\|\boldsymbol{v}_n\|_{1,q,\mathcal{M}_n} \lesssim h_n|\boldsymbol{v}|_{\mathbf{W}^{1,q}(\Omega)^3}$. Combining this with (5.16) yields the result. ∎

We can now give the proof of Proposition (5.4).

Proof of Proposition 5.4. To prove this result, we pass to the limit $n \to +\infty$ in the weak formulation of the discrete momentum balance. Let $\boldsymbol{v} \in \mathcal{C}_c^\infty(\Omega)^3$ and for $n \in \mathbb{N}$, define $\boldsymbol{v}_n = I_{\mathcal{M}_n}\boldsymbol{v} \in \mathbf{H}_{\mathcal{M}_n,0}(\Omega)$. We have $\|\boldsymbol{v}_n\|_{1,q,\mathcal{M}_n} \leq C\,\|\boldsymbol{v}\|_{\mathrm{W}_0^{1,q}(\Omega)^3}$ for all $q \in [1, +\infty)$ by Lemma 5.5. Taking the test function

$\boldsymbol{v}_n$ in the weak formulation of the discrete momentum balance (4.20), we get for all $n \in \mathbb{N}$:

$$
\begin{aligned}
- \int_\Omega (\mathcal{P}_{\mathcal{E}_n} \rho_n) \, (\Pi_{\mathcal{E}_n} \boldsymbol{u}_n) \otimes (\Pi_{\mathcal{E}_n} \boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} \\
+ \mu \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{u}_n : \boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} + (\mu + \lambda) \int_\Omega \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n \, \mathrm{div}_{\mathcal{M}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} \\
- a \int_\Omega \rho_n^\gamma \, \mathrm{div}_{\mathcal{M}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega h_n^{\xi_3} \rho_n^\Gamma \, \mathrm{div}_{\mathcal{M}_n} \boldsymbol{v}_n + R_{\mathrm{conv}}(\rho_n, \boldsymbol{u}_n, \boldsymbol{v}_n) = \int_\Omega \boldsymbol{f} \cdot \Pi_{\mathcal{E}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{5.20}
$$

The term involving the artificial pressure tends to zero as $n \to +\infty$ since $(h_n^{\xi_3} \rho_n^\Gamma)_{n\in\mathbb{N}}$ converges strongly to 0 in $\mathrm{L}^p(\Omega)$ for some $1 < p < 1 + \eta$ (see (5.4)) and $(\mathrm{div}_{\mathcal{M}_n} \boldsymbol{v}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^q(\Omega)$ for all $q \in (1, +\infty)$. On the other hand, Lemma 4.13 gives

$$
\begin{aligned}
\big| R_{\mathrm{conv}}(\rho_n, \boldsymbol{u}_n, \boldsymbol{v}_n) \big| \leq C \, h_n^{\frac{1}{2} - \frac{1}{\Gamma}\left(\frac{3}{1+\eta} + \xi_3\right)} \| h_n^{\xi_3} \rho^\Gamma \|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\Gamma}} \| \boldsymbol{u}_n \|_{1,2,\mathcal{M}_n}^2 \, \| \boldsymbol{v}_n \|_{1,2,\mathcal{M}_n} \\
+ C \, h_n^{\xi_2 - \frac{1}{\eta} - \frac{1}{\eta\Gamma}\left(\frac{3}{1+\eta} + \xi_3\right)} \| h_n^{\xi_3} \rho_n^\Gamma \|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\eta\Gamma}} \| \boldsymbol{u}_n \|_{1,2,\mathcal{M}_n} \, \| \boldsymbol{v}_n \|_{1,2,\mathcal{M}_n},
\end{aligned}
$$

with C independent of n, so $R_{\mathrm{conv}}(\rho_n, \boldsymbol{u}_n, \boldsymbol{v}_n) \to 0$ as $n \to +\infty$ using Remark 4.6. We also easily obtain the convergence of the diffusion and pressure terms. Since $(\boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{u}_n)_{n\in\mathbb{N}}$ (*resp.* $(\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n)_{n\in\mathbb{N}}$) weakly converges to $\boldsymbol{\nabla} \boldsymbol{u}$ (*resp.* $\mathrm{div}\, \boldsymbol{u}$) in $\mathbf{L}^2(\Omega)^3$, $(\rho_n^\gamma)_{n\in\mathbb{N}}$ weakly converges to $\overline{\rho^\gamma}$ in $\mathrm{L}^{1+\eta}(\Omega)$ and $(\boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{v}_n)_{n\in\mathbb{N}}$ (*resp.* $(\mathrm{div}_{\mathcal{M}_n} \boldsymbol{v}_n)_{n\in\mathbb{N}}$)) strongly converges to $\boldsymbol{\nabla} \boldsymbol{v}$ (*resp.* $\mathrm{div}\, \boldsymbol{v}$) in $\mathbf{L}^q(\Omega)^3$ for all $q \in (1, +\infty)$ we obtain:

$$
\begin{aligned}
\lim_{n \to +\infty} \Big(\mu \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{u}_n : \boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} + (\mu + \lambda) \int_\Omega \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n \, \mathrm{div}_{\mathcal{M}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} \\
- a \int_\Omega \rho_n^\gamma \, \mathrm{div}_{\mathcal{M}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \Pi_{\mathcal{E}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} \Big) \\
= \mu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + (\mu + \lambda) \int_\Omega \mathrm{div}\, \boldsymbol{u} \, \mathrm{div}\, \boldsymbol{v} \; \mathrm{d}\boldsymbol{x} - a \int_\Omega \overline{\rho^\gamma} \, \mathrm{div}\, \boldsymbol{v} \; \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

The convergence of the source term is given by (5.18).

Let us now prove the convergence of the convective term. We have:

$$
\begin{aligned}
- \int_\Omega (\mathcal{P}_{\mathcal{E}_n} \rho_n) \, (\Pi_{\mathcal{E}_n} \boldsymbol{u}_n) \otimes (\Pi_{\mathcal{E}_n} \boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n} \boldsymbol{v}_n \; \mathrm{d}\boldsymbol{x} \\
= - \sum_{\substack{\sigma \in \mathcal{E}_{n,\mathrm{int}} \\ \sigma = K|L}} |\sigma| \, \rho_\sigma \, \boldsymbol{u}_\sigma \otimes \boldsymbol{u}_\sigma : (\boldsymbol{v}_L - \boldsymbol{v}_K) \otimes \boldsymbol{n}_{K,\sigma} \\
= - \sum_{\substack{\sigma \in \mathcal{E}_{n,\mathrm{int}} \\ \sigma = K|L}} |\sigma| \, \tfrac{1}{2}(\rho_K + \rho_L) \, \boldsymbol{u}_\sigma \otimes \boldsymbol{u}_\sigma : (\boldsymbol{v}_L - \boldsymbol{v}_K) \otimes \boldsymbol{n}_{K,\sigma} + R_1^n,
\end{aligned}
\tag{5.21}
$$

with

$$
R_1^n = \sum_{\substack{\sigma \in \mathcal{E}_{n,\mathrm{int}} \\ \sigma = K|L}} |\sigma| \, \big(\tfrac{1}{2}(\rho_K + \rho_L) - \rho_\sigma \big) \, \boldsymbol{u}_\sigma \otimes \boldsymbol{u}_\sigma : (\boldsymbol{v}_L - \boldsymbol{v}_K) \otimes \boldsymbol{n}_{K,\sigma}.
$$

Reordering the sum in the first term of (5.21) we get:

$$
\begin{aligned}
-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \,\mathrm{d}\boldsymbol{x}
&= -\frac{1}{2}\sum_{K\in\mathcal{M}_n}\rho_K \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\,\boldsymbol{u}_\sigma\otimes\boldsymbol{u}_\sigma : (\boldsymbol{v}_L-\boldsymbol{v}_K)\otimes\boldsymbol{n}_{K,\sigma} + R_1^n \\
&= -\frac{1}{2}\sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\otimes\boldsymbol{u}_K : \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\,(\boldsymbol{v}_L-\boldsymbol{v}_K)\otimes\boldsymbol{n}_{K,\sigma} + R_1^n + R_2^n, \qquad (5.22)
\end{aligned}
$$

where $\boldsymbol{u}_K$ is the mean value of $\boldsymbol{u}_n$ over K and

$$
R_2^n = \frac{1}{2}\sum_{K\in\mathcal{M}_n}\rho_K \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\,(\boldsymbol{u}_K\otimes\boldsymbol{u}_K - \boldsymbol{u}_\sigma\otimes\boldsymbol{u}_\sigma) : (\boldsymbol{v}_L-\boldsymbol{v}_K)\otimes\boldsymbol{n}_{K,\sigma}.
$$

Back to (5.22) we get:

$$
\begin{aligned}
-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \,\mathrm{d}\boldsymbol{x}
&= -\sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\otimes\boldsymbol{u}_K : \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\,\tfrac{1}{2}(\boldsymbol{v}_L+\boldsymbol{v}_K)\otimes\boldsymbol{n}_{K,\sigma} + R_1^n + R_2^n + R_3^n \\
&= -\sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\otimes\boldsymbol{u}_K : \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\boldsymbol{v}_\sigma\otimes\boldsymbol{n}_{K,\sigma} + R_1^n + R_2^n + R_3^n + R_4^n \\
&= -\int_\Omega \rho_n\,\boldsymbol{u}_n\otimes\boldsymbol{u}_n : \overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\boldsymbol{v}_n \,\mathrm{d}\boldsymbol{x} + R_1^n + R_2^n + R_3^n + R_4^n,
\end{aligned}
$$

with

$$
\begin{aligned}
R_3^n &= \sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\otimes\boldsymbol{u}_K : \boldsymbol{v}_K\otimes\Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\boldsymbol{n}_{K,\sigma}\Big), \\
R_4^n &= \sum_{K\in\mathcal{M}_n}\rho_K\,\boldsymbol{u}_K\otimes\boldsymbol{u}_K : \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma|\,(\boldsymbol{v}_\sigma - \tfrac{1}{2}(\boldsymbol{v}_L+\boldsymbol{v}_K))\otimes\boldsymbol{n}_{K,\sigma}.
\end{aligned}
$$

Since $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\mathbf{L}^q(\Omega)$ for all $q\in[1,6)$, and $\overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\boldsymbol{v}_n \to \boldsymbol{\nabla}\boldsymbol{v}$ in $\mathbf{L}^r(\Omega)^3$ for all $r\in(1,+\infty)$, we have $\boldsymbol{u}_n\otimes\boldsymbol{u}_n : \overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\boldsymbol{v}_n \to \boldsymbol{u}\otimes\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v}$ in $\mathrm{L}^{3-\delta}(\Omega)$ for all $\delta\in(0,2]$. Furthermore, we have $\rho_n \rightharpoonup \rho$ weakly in $\mathrm{L}^{3(\gamma-1)}(\Omega)$ with $3(\gamma-1) > \frac{3}{2}$ (since $\gamma > \frac{3}{2}$), which yields:

$$
\lim_{n\to+\infty} -\int_\Omega \rho_n\,\boldsymbol{u}_n\otimes\boldsymbol{u}_n : \overline{\boldsymbol{\nabla}}_{\mathcal{M}_n}\boldsymbol{v}_n \,\mathrm{d}\boldsymbol{x} = -\int_\Omega \rho\,\boldsymbol{u}\otimes\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v} \,\mathrm{d}\boldsymbol{x}.
$$

In order to complete the proof of Proposition 5.4, it remains to prove that $\sum_{i=1}^4 R_i^n \to 0$ as $n\to+\infty$. In the following, in order to ease the notations, we denote $A_n \lesssim B_n$ when there is a constant C, independent of n, such that $A_n \le C\,B_n$. We begin with R_1^n. Recalling the upwind definition of ρ_σ

and the fact that $\boldsymbol{a} \otimes \boldsymbol{b} : \boldsymbol{c} \otimes \boldsymbol{d} = (\boldsymbol{a} \cdot \boldsymbol{c})\,(\boldsymbol{b} \cdot \boldsymbol{d})$ for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$ we get:

$$|R_1^n| \leq \frac{1}{2} \sum_{\substack{\sigma \in \mathcal{E}_{n,\text{int}} \\ \sigma = K|L}} |\sigma|\,|\rho_K - \rho_L|\,|\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}|\,|\boldsymbol{v}_L - \boldsymbol{v}_K|\,|\boldsymbol{u}_\sigma|.$$

Applying the Cauchy-Schwarz inequality, we get:

$$\begin{aligned}
|R_1^n| &\leq \frac{1}{2} \Big(\sum_{\substack{\sigma \in \mathcal{E}_{n,\text{int}} \\ \sigma = K|L}} |\sigma|\,|\rho_K - \rho_L|^2\,|\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}| \Big)^{\frac{1}{2}} \Big(\sum_{\substack{\sigma \in \mathcal{E}_{n,\text{int}} \\ \sigma = K|L}} |\sigma|\,|\boldsymbol{v}_L - \boldsymbol{v}_K|^2\,|\boldsymbol{u}_\sigma|^3 \Big)^{\frac{1}{2}} \\
&\lesssim h_n^{-\frac{5}{8\Gamma}\left(\frac{3}{1+\eta} + \xi_3\right)} \Big(\sum_{\substack{\sigma \in \mathcal{E}_{n,\text{int}} \\ \sigma = K|L}} |\sigma|\,|\boldsymbol{v}_L - \boldsymbol{v}_K|^2\,|\boldsymbol{u}_\sigma|^3 \Big)^{\frac{1}{2}}
\end{aligned}$$

by estimate (5.4). By Taylor's inequality applied to the smooth function $\boldsymbol{v}$ and the regularity of the discretization, we have $|\boldsymbol{v}_L - \boldsymbol{v}_K|^2 \lesssim h_n\,|D_\sigma|/|\sigma|\,\|\boldsymbol{\nabla v}\|^2_{\mathbf{L}^\infty(\Omega)^3}$. Hence:

$$|R_1^n| \lesssim h_n^{\frac{5}{8\Gamma}\left(\frac{4}{5}\Gamma - \frac{3}{1+\eta} - \xi_3\right)} \|\Pi_{\mathcal{E}_n} \boldsymbol{u}_n\|^{\frac{3}{2}}_{\mathbf{L}^3(\Omega)} \lesssim h_n^{\frac{5}{8\Gamma}\left(\frac{4}{5}\Gamma - \frac{3}{1+\eta} - \xi_3\right)} \|\boldsymbol{u}_n\|^{\frac{3}{2}}_{\mathbf{L}^6(\Omega)} \lesssim h_n^{\frac{5}{8\Gamma}\left(\frac{4}{5}\Gamma - \frac{3}{1+\eta} - \xi_3\right)}$$

since $\|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)}$ is controlled by $\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n}$ which is bounded by C_0. Since (Γ, ξ_3) satisfy (5.2), we get $R_1^n \to 0$ as $n \to +\infty$. We now turn to R_2^n. We write

$$\boldsymbol{u}_K \otimes \boldsymbol{u}_K - \boldsymbol{u}_\sigma \otimes \boldsymbol{u}_\sigma = (\boldsymbol{u}_K - \boldsymbol{u}_\sigma) \otimes \boldsymbol{u}_K + \boldsymbol{u}_\sigma \otimes (\boldsymbol{u}_K - \boldsymbol{u}_\sigma).$$

Hence, $|R_2^n| \leq |R_{2,1}^n| + |R_{2,2}^n|$ with:

$$\begin{aligned}
|R_{2,1}^n| &= \frac{1}{2} \sum_{K \in \mathcal{M}_n} \rho_K \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma|\,|\boldsymbol{u}_K - \boldsymbol{u}_\sigma|\,|\boldsymbol{u}_K|\,|\boldsymbol{v}_L - \boldsymbol{v}_K|, \\
|R_{2,2}^n| &= \frac{1}{2} \sum_{K \in \mathcal{M}_n} \rho_K \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma|\,|\boldsymbol{u}_K - \boldsymbol{u}_\sigma|\,|\boldsymbol{u}_\sigma|\,|\boldsymbol{v}_L - \boldsymbol{v}_K|.
\end{aligned}$$

We only treat $|R_{2,1}^n|$, since the treatment of $|R_{2,2}^n|$ is similar. By a Taylor inequality on the smooth function $\boldsymbol{v}$ and the regularity of the discretization, we have: $|\boldsymbol{v}_L - \boldsymbol{v}_K| \lesssim h_n\,\|\boldsymbol{\nabla v}\|_{\mathbf{L}^\infty(\Omega)^3}$. Hence:

$$|R_{2,1}^n| \lesssim h_n \sum_{K \in \mathcal{M}_n} \rho_K\,|\boldsymbol{u}_K| \sum_{\sigma \in \mathcal{E}(K)} |\sigma|\,|\boldsymbol{u}_K - \boldsymbol{u}_\sigma|. \tag{5.23}$$

Proceeding as in the proof of Proposition 5.2 (see the computation after (5.12)) we get:

$$\begin{aligned}
|R_{2,1}^n| &\lesssim h_n \sum_{K \in \mathcal{M}_n} |K|^{\frac{1}{2}} \rho_K\,|\boldsymbol{u}_K|\,\|\boldsymbol{\nabla u}_n\|_{\mathbf{L}^2(K)^3} \\
&\lesssim h_n\,\|\rho_n\|^{1 - \frac{3(\gamma-1)}{2}}_{\mathrm{L}^\infty(\Omega)}\,\|\boldsymbol{u}_n\|_{\mathbf{L}^\infty(\Omega)} \sum_{K \in \mathcal{M}_n} |K|^{\frac{1}{2}} \rho_K^{\frac{3(\gamma-1)}{2}}\,\|\boldsymbol{\nabla u}_n\|_{\mathbf{L}^2(K)^3} \\
&\lesssim h_n\,\|\rho_n\|^{\frac{5-3\gamma}{2}}_{\mathrm{L}^\infty(\Omega)}\,\|\boldsymbol{u}_n\|_{\mathbf{L}^\infty(\Omega)}.
\end{aligned}$$

Recalling inverse inequalities $\|\rho_n\|_{\mathrm{L}^\infty(\Omega)} \lesssim h_n^{-\frac{1}{\gamma-1}} \|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)}$ and $\|\boldsymbol{u}_n\|_{\mathbf{L}^\infty(\Omega)} \lesssim h_n^{-\frac{1}{2}} \|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)}$, and since $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{3(\gamma-1)}(\Omega)$ and $(\|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)})_{n\in\mathbb{N}}$ is bounded we get:

$$|R^n_{2,1}| \lesssim h_n^{1-\frac{1}{\gamma-1}\frac{5-3\gamma}{2}-\frac{1}{2}} = h_n^{\frac{2\gamma-3}{\gamma-1}}.$$

Since, $\gamma > \frac{3}{2}$, we get $R^n_{2,1} \to 0$ as $n \to +\infty$. As said previously, the same holds for $R^n_{2,2}$. The third remainder term satisfies $R^n_3 = 0$ since $\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, \boldsymbol{n}_{K,\sigma} = 0$ for all $K \in \mathcal{M}_n$. Let us conclude with the control of R^n_4. Denoting $\hat{\boldsymbol{v}}_\sigma = \frac{1}{2}(\boldsymbol{v}_L + \boldsymbol{v}_K)$ for $\sigma = K|L$, we may write $R^n_4 = R^n_{4,1} + R^n_{4,2}$ with:

$$R^n_{4,1} = \sum_{K\in\mathcal{M}_n} \rho_K \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, (\boldsymbol{u}_K \otimes \boldsymbol{u}_K - \boldsymbol{u}_\sigma \otimes \boldsymbol{u}_\sigma) : (\boldsymbol{v}_\sigma - \hat{\boldsymbol{v}}_\sigma) \otimes \boldsymbol{n}_{K,\sigma} \Big),$$
$$R^n_{4,2} = \sum_{K\in\mathcal{M}_n} \rho_K \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\, \boldsymbol{u}_\sigma \otimes \boldsymbol{u}_\sigma : (\boldsymbol{v}_\sigma - \hat{\boldsymbol{v}}_\sigma) \otimes \boldsymbol{n}_{K,\sigma} \Big).$$

The term $R^n_{4,1}$ can be controlled in the same way as R^n_2 and we obtain $R^n_{4,1} \to 0$ as $n \to +\infty$. Reordering the sum in $R^n_{4,2}$ we get:

$$R^n_{4,2} = \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}} \\ \sigma=K|L}} |\sigma|\, (\rho_K - \rho_L)\, \boldsymbol{u}_\sigma \otimes \boldsymbol{u}_\sigma : (\boldsymbol{v}_\sigma - \hat{\boldsymbol{v}}_\sigma) \otimes \boldsymbol{n}_{K,\sigma}.$$

Hence $R^n_{4,2}$ can be controlled in the same way as R^n_1 and we obtain $R^n_{4,2} \to 0$ as $n \to +\infty$. This concludes the proof of (5.13).

It remains to prove (5.14). We proceed as in the proof of Proposition 4.11. Taking $\boldsymbol{u}_n$ as a test function in the first form of the discrete weak formulation of the momentum equation and using (4.13) with $\beta = \gamma$ and $\beta = \Gamma$ we get:

$$\frac{1}{2} h_n^{\xi_1} \int_\Omega (\tilde{\rho}_n - \rho^\star)\, |\Pi_{\mathcal{E}_n} \boldsymbol{u}_n|^2 \,\mathrm{d}\boldsymbol{x} + \mu \int_\Omega |\boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{u}_n|^2 \,\mathrm{d}\boldsymbol{x}$$
$$+ (\mu + \lambda) \int_\Omega (\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n)^2 \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \boldsymbol{f} \cdot \Pi_{\mathcal{E}_n} \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x}, \qquad \forall n \in \mathbb{N},$$

where $\tilde{\rho}_n$ is the piecewise constant scalar function which is equal to ρ_{D_σ} on every dual cell D_σ, and which satisfies $\tilde{\rho}_n > 0$ (because $\rho_n > 0$) and $\int_\Omega \tilde{\rho}_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho_n \,\mathrm{d}\boldsymbol{x} = |\Omega|\rho^\star$. Since $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{\frac{3}{2}}(\Omega)$ and $(\Pi_{\mathcal{E}_n} \boldsymbol{u}_n)_{n\in\mathbb{N}}$ in $\mathbf{L}^6(\Omega)$, the first term tends to zero as $n \to +\infty$. Thus, passing to the limit $n \to +\infty$ in the above inequality and recalling that $\boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{u}_n \rightharpoonup \boldsymbol{\nabla} \boldsymbol{u}$ weakly in $\mathbf{L}^2(\Omega)^3$ and $\Pi_{\mathcal{E}_n} \boldsymbol{u}_n \to \boldsymbol{u}$ strongly in (say) $\mathbf{L}^2(\Omega)$ yields (5.14).

□

5.3 Passing to the limit in the equation of state

5.3.1 Weak compactness of the effective viscous flux

As in the continuous case, the equation of state is satisfied at the limit as a consequence of the convergence of the so-called effective viscous flux. Indeed, we have the following result.

Proposition 5.7. *Under the assumptions of Theorem 3.1, let* $(\rho, \boldsymbol{u}, \overline{\rho^\gamma}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}(\Omega)$ *be the limit triple of the sequence* $(\rho_n, \boldsymbol{u}_n, \rho_n^\gamma)_{n\in\mathbb{N}}$*. For* $k \in \mathbb{N}^*$*, define*

$$T_k(t) = \begin{cases} t & \text{if } t \in [0,k), \\ k & \text{if } t \in [k, +\infty). \end{cases}$$

The sequence $(T_k(\rho_n))_{n\in\mathbb{N}}$ *is bounded in* $\mathrm{L}^\infty(\Omega)$ *and, up to extracting a subsequence, it converges for the weak-* topology in* $\mathrm{L}^\infty(\Omega)$ *towards some function denoted* $\overline{T_k(\rho)}$*. Then (up to extracting a subsequence) the following identity holds:*

$$\lim_{n\to+\infty} \int_\Omega \big((2\mu+\lambda)\ \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n - a\rho_n^\gamma\big) T_k(\rho_n)\phi \,\mathrm{d}\boldsymbol{x} \\ = \int_\Omega \big((2\mu+\lambda)\,\mathrm{div}\,\boldsymbol{u} - a\overline{\rho^\gamma}\big)\overline{T_k(\rho)}\phi \,\mathrm{d}\boldsymbol{x}, \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega).$$

Remark 5.1. *As in the continuous case, this result is obtained by taking the test function* $\boldsymbol{v} = \phi\widetilde{\boldsymbol{w}}_n$ *in the discrete momentum equation (4.20), where* $\widetilde{\boldsymbol{w}}_n$ *is computed from* $T_k(\rho_n)$ *by applying Lemma 2.8,* i.e. $\widetilde{\boldsymbol{w}}_n = \mathcal{A}T_k(\rho_n)$*, and satisfies* $\mathrm{div}\,\widetilde{\boldsymbol{w}}_n = T_k(\rho_n)$*,* $\mathrm{curl}\,\widetilde{\boldsymbol{w}}_n = 0$*. Unfortunately, the discrete gradient, divergence and rotational operators associated with the Crouzeix-Raviart approximation do not satisfy a discrete equivalent of the global identity (2.19), namely*

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{curl}\,\boldsymbol{u}\,\mathrm{curl}\,\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}.$$

Instead, one needs to apply (2.18) locally on each control volume $K \in \mathcal{M}_n$*. The accumulating boundary terms must then be controlled through an estimate of* $\widetilde{\boldsymbol{w}}_n$ *in* $\mathbf{W}^{2,2}(\Omega)$*. Moreover, it also appears in the analysis that the control of some remainder terms involving the pressure (which is controlled in* $\mathrm{L}^{1+\eta}(\Omega)$*) requires an estimate of* $\widetilde{\boldsymbol{w}}_n$ *in* $\mathbf{W}^{2,\frac{1+\eta}{\eta}}(\Omega)$*. Since* $\frac{1+\eta}{\eta} \geq 2$*, this latter control is more restrictive. Such control is what motivates the introduction of the stabilization term*

$$T^2_{\mathrm{stab}} = -h_{\mathcal{M}}^{\xi_2} \Delta_{\frac{1+\eta}{\eta},\mathcal{M}}(\rho)$$

in the numerical scheme. For the MAC scheme, studied for instance in [22], we directly have an equivalent of (2.19) and T^2_{stab} *is useless.*

The function $\widetilde{\boldsymbol{w}}_n$ defined in Remark 5.1 is not in $\mathbf{W}^{2,\frac{1+\eta}{\eta}}(\Omega)$ because $T_k(\rho_n)$ is not in $\mathrm{W}^{1,\frac{1+\eta}{\eta}}(\Omega)$. We rather define $\boldsymbol{w}_n = \mathcal{A}(i_{\mathcal{M}_n} T_k(\rho_n))$ so that $\mathrm{div}\,\boldsymbol{w}_n = i_{\mathcal{M}_n} T_k(\rho_n)$ and $\mathrm{curl}\,\boldsymbol{w}_n = 0$ where $i_{\mathcal{M}}\, T_k(\rho)$ is a regularization of $T_k(\rho)$, the $\mathrm{W}^{1,\frac{1+\eta}{\eta}}$ semi-norm of which is controlled by $|\rho|_{\frac{1+\eta}{\eta},\mathcal{M}}$. The operator $i_{\mathcal{M}}$ is specified in the following definition and its properties in Lemma 5.8.

Definition 5.1. *Let* $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ *be a staggered discretization of* Ω *and* $\mathcal{S}$ *be the set of vertices of the primal mesh* $\mathcal{M}$*. For* $s \in \mathcal{S}$*, we denote by* $\mathcal{N}_s \subset \mathcal{M}$ *the set of the elements* $K \in \mathcal{M}$ *of which* s *is a vertex. Let* $p \in \mathrm{L}_{\mathcal{M}}(\Omega)$*. We denote* $i_{\mathcal{M}}\, p$ *the function defined as follows:*

- $i_{\mathcal{M}}\, p \in \mathcal{C}^0(\Omega)$*,*

- *for all $K \in \mathcal{M}$, the restriction of $i_{\mathcal{M}}\, p$ to K is affine,*
- *for all $s \in \mathcal{S}$, $(i_{\mathcal{M}}\, p)(s) = \dfrac{1}{\mathrm{card}(\mathcal{N}_s)} \displaystyle\sum_{K \in \mathcal{N}_s} p_K$.*

Lemma 5.8. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \le \theta_0$, with $\theta_{\mathcal{M}}$ defined by (3.1). For all $r \in [1, +\infty]$ there exists $C = C(r, \theta_0)$ such that:*

$$\|i_{\mathcal{M}}\, p\|_{\mathrm{L}^r(\Omega)} \le C \|p\|_{\mathrm{L}^r(\Omega)}, \qquad \forall p \in \mathrm{L}_{\mathcal{M}}(\Omega). \tag{5.24}$$

Moreover, for all $p \in \mathrm{L}_{\mathcal{M}}(\Omega)$ we have $i_{\mathcal{M}}\, p \in \mathrm{W}^{1,q}(\Omega)$ for all $q \in [1, +\infty)$ and there exists a constant $C = C(q, \theta_0)$ such that :

$$\|i_{\mathcal{M}}\, p - p\|_{\mathrm{L}^q(\Omega)} + h_{\mathcal{M}}\, |i_{\mathcal{M}}\, p|_{\mathrm{W}^{1,q}(\Omega)} \le C\, h_{\mathcal{M}}\, |p|_{q,\mathcal{M}} \qquad \forall p \in \mathrm{L}_{\mathcal{M}}(\Omega). \tag{5.25}$$

Proof. The proof is similar to that of [13, Lemma 5.8]. We skip the details. □

We also have the following technical result which will be useful hereinafter. The proof can be found in [20, Lemma 2.4].

Lemma 5.9. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. Let $(n_\sigma)_{\sigma \in \mathcal{E}_{\mathrm{int}}}$ be a family of real numbers such that for all $\sigma \in \mathcal{E}_{\mathrm{int}}$, $|n_\sigma| \le 1$, and let $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$. Then, for any $q \in (1, \infty)$ there exists $C = C(q, \theta_0)$ such that:*

$$\sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \left| \int_\sigma n_\sigma\, [\boldsymbol{u}]_\sigma\, \boldsymbol{f}\, \mathrm{d}\sigma(\boldsymbol{x}) \right| \le C\, h_{\mathcal{M}}\, \|\boldsymbol{u}\|_{1,q',\mathcal{M}}\, |\boldsymbol{f}|_{\mathrm{W}^{1,q}(\Omega)}, \qquad \forall \boldsymbol{f} \in \mathrm{W}^{1,q}_0(\Omega),$$

where $q' = \frac{q-1}{q}$, $\|\boldsymbol{u}\|^2_{1,q',\mathcal{M}} = \sum_{K \in \mathcal{M}} \int_K |\boldsymbol{\nabla}\, \boldsymbol{u}|^{q'}\, \mathrm{d}\boldsymbol{x}$.

We may now give the proof Proposition 5.7 which is similar to that of [13, Prop. 5.9 and 5.10]. The main difference is that we here have to handle the additional convective term in the momentum balance.

Proof of Proposition 5.7. Let $k \in \mathbb{N}^*$. Since $(T_k(\rho_n))_{n \in \mathbb{N}}$ is bounded in $\mathrm{L}^\infty(\Omega)$ (by k) we have by (5.24):

$$\|i_{\mathcal{M}_n}\, T_k(\rho_n)\|_{\mathrm{L}^\infty(\Omega)} \lesssim 1. \tag{5.26}$$

Furthermore, by (5.25) and (5.4) (observing that $|T_k(r_1) - T_k(r_2)| \le |r_1 - r_2|$ for all $r_1, r_2 \ge 0$) we have for all $n \in \mathbb{N}$:

$$\|i_{\mathcal{M}_n}\, T_k(\rho_n)\|_{\mathrm{W}^{1,\frac{1+\eta}{\eta}}(\Omega)} \lesssim h_n^{-\xi}, \quad \|i_{\mathcal{M}_n}\, T_k(\rho_n) - T_k(\rho_n)\|_{\mathrm{L}^{\frac{1+\eta}{\eta}}(\Omega)} \lesssim h_n^{1-\xi}, \tag{5.27}$$

where, by assumption (5.3) on ξ_2,

$$\xi = \frac{\eta}{1+\eta}\left[\xi_2 + \frac{5}{4\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)\right] < 1. \tag{5.28}$$

Let $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ be the sequence of functions defined from $(i_{\mathcal{M}_n}\, T_k(\rho_n))_{n\in\mathbb{N}}$ by Lemma 2.8. We have

$$\mathrm{div}\ \boldsymbol{w}_n = i_{\mathcal{M}_n}\, T_k(\rho_n), \quad \mathrm{curl}\ \boldsymbol{w}_n = 0, \quad \|\boldsymbol{w}_n\|_{\mathbf{W}^{1,q}(\Omega)} \lesssim 1, \quad \forall q \in (1,+\infty).$$

Moreover, by the Sobolev injection $\mathrm{W}^{1,q}(\Omega) \subset \mathrm{L}^\infty(\Omega)$ for $q > 3$, the sequence $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^\infty(\Omega)$ and up to extracting a subsequence, as $n \to +\infty$, it strongly converges in $\mathbf{L}^q(\Omega)$ and weakly in $\mathbf{W}^{1,q}(\Omega)$ for all $q \in (1,+\infty)$ towards some function $\boldsymbol{w}$ satisfying:

$$\mathrm{div}\ \boldsymbol{w} = \overline{T_k(\rho)} \quad \text{and} \quad \mathrm{curl}\ \boldsymbol{w} = 0.$$

Inequality (5.27) and the properties of operator $\mathcal{A}$ yield

$$\|\boldsymbol{w}_n\|_{\mathbf{W}^{2,\frac{1+\eta}{\eta}}(\Omega)} \lesssim \|i_{\mathcal{M}_n}\, T_k(\rho_n)\|_{\mathrm{W}^{1,\frac{1+\eta}{\eta}}(\Omega)} \lesssim h_n^{-\xi}. \tag{5.29}$$

Let $\phi \in \mathcal{C}_c^\infty(\Omega)$ and take $\boldsymbol{v}_n = I_{\mathcal{M}_n}(\phi \boldsymbol{w}_n) \in \mathbf{H}_{\mathcal{M}_n,0}(\Omega)$ as a test function in the discrete weak formulation of the momentum balance (4.20). We get for all $n \in \mathbb{N}$:

$$\begin{aligned} -\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} \\ + \mu \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}_n}\boldsymbol{u}_n : \boldsymbol{\nabla}_{\mathcal{M}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} + (\mu+\lambda)\int_\Omega \mathrm{div}_{\mathcal{M}_n}\boldsymbol{u}_n \,\mathrm{div}_{\mathcal{M}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} \\ - a\int_\Omega \rho_n^\gamma \,\mathrm{div}_{\mathcal{M}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} + R_1^n = \int_\Omega \boldsymbol{f}\cdot \Pi_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{5.30}$$

where

$$R_1^n = -\int_\Omega h_n^{\xi_3}\rho_n^\Gamma \,\mathrm{div}_{\mathcal{M}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} + R_{\mathrm{conv}}(\rho_n,\boldsymbol{u}_n,\boldsymbol{v}_n).$$

By Lemma 4.13, we have:

$$\begin{aligned} \big|R_{\mathrm{conv}}(\rho_n,\boldsymbol{u}_n,\boldsymbol{v}_n)\big| \le C\, h_n^{\frac{1}{2}-\frac{1}{\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \|h_n^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\Gamma}} \|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n}^2 \|\boldsymbol{v}_n\|_{1,2,\mathcal{M}_n} \\ + C\, h_n^{\xi_2-\frac{1}{\eta}-\frac{1}{\eta\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \|h_n^{\xi_3}\rho_n^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\eta\Gamma}} \|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n} \|\boldsymbol{v}_n\|_{1,2,\mathcal{M}_n}. \end{aligned}$$

Since $\|\boldsymbol{v}_n\|_{1,2,\mathcal{M}_n} \lesssim \|\phi\boldsymbol{w}_n\|_{\mathbf{H}^1(\Omega)}$, we can apply Remark 4.6 and we get that $\big|R_{\mathrm{conv}}(\rho_n,\boldsymbol{u}_n,\boldsymbol{v}_n)\big| \to 0$ as $n \to +\infty$. Moreover, by (5.4), we have $h_n^{\xi_3}\rho_n^\Gamma \to 0$ in $\mathrm{L}^p(\Omega)$ with $1 < p < 1+\eta$ as $n \to +\infty$. Since $(\mathrm{div}_{\mathcal{M}_n}\boldsymbol{v}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{p'}(\Omega)$, we obtain that $R_1^n \to 0$ as $n \to +\infty$. Hence, denoting $\boldsymbol{\delta}_n = \boldsymbol{v}_n - \phi\boldsymbol{w}_n = I_{\mathcal{M}_n}(\phi\boldsymbol{w}_n) - \phi\boldsymbol{w}_n$, we have

$$- \int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x}$$
$$+ \mu \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}_n}\boldsymbol{u}_n : \boldsymbol{\nabla}(\phi \boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x} + (\mu+\lambda)\int_\Omega \mathrm{div}_{\mathcal{M}_n}\boldsymbol{u}_n\, \mathrm{div}(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x}$$
$$- a \int_\Omega \rho_n^\gamma\, \mathrm{div}(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x} + R_2^n + \underset{n\to+\infty}{o}(1) = \int_\Omega \boldsymbol{f}\cdot(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x}, \quad (5.31)$$

where:

$$R_2^n = \mu \int_\Omega \boldsymbol{\nabla}_{\mathcal{M}_n}\boldsymbol{u}_n : \boldsymbol{\nabla}_{\mathcal{M}_n}\boldsymbol{\delta}_n\,\mathrm{d}\boldsymbol{x} + (\mu+\lambda)\int_\Omega \mathrm{div}_{\mathcal{M}_n}\boldsymbol{u}_n\,\mathrm{div}_{\mathcal{M}_n}\boldsymbol{\delta}_n\,\mathrm{d}\boldsymbol{x}$$
$$- a\int_\Omega \rho_n^\gamma\,\mathrm{div}_{\mathcal{M}_n}\boldsymbol{\delta}_n\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\delta}_n\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}\cdot(\Pi_{\mathcal{E}_n}\boldsymbol{v}_n - \boldsymbol{v}_n)\,\mathrm{d}\boldsymbol{x}.$$

By the properties of the Fortin operator $I_{\mathcal{M}_n}$, we have $\|\boldsymbol{\delta}_n\|_{\mathrm{L}^2(\Omega)} \lesssim h_n^2|\phi\boldsymbol{w}_n|_{\mathbf{W}^{2,\frac{1+\eta}{\eta}}(\Omega)} \lesssim h_n^{2-\xi}$ and $\|\boldsymbol{\delta}_n\|_{1,\frac{1+\eta}{\eta},\mathcal{M}_n} \lesssim h_n|\phi\boldsymbol{w}_n|_{\mathbf{W}^{2,\frac{1+\eta}{\eta}}(\Omega)} \lesssim h_n^{1-\xi}$ with $\frac{1+\eta}{\eta} \geq 2$. Since $(\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n})_{n\in\mathbb{N}}$ is bounded, $(\rho_n^\gamma)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{1+\eta}(\Omega)$ (recall that $1+\eta = \frac{3(\gamma-1)}{\gamma}$), $\Pi_{\mathcal{E}_n}\boldsymbol{v}_n - \boldsymbol{v}_n \to 0$ in $\mathbf{L}^2(\Omega)$ as $n\to+\infty$, and $\xi < 1$, we get that $R_2^n \to 0$ as $n \to +\infty$.

Applying the identity (2.18) over each control volume, we get:

$$\mu\int_\Omega \boldsymbol{\nabla}_{\mathcal{M}_n}\boldsymbol{u}_n : \boldsymbol{\nabla}(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x}$$
$$= \mu\int_\Omega \mathrm{div}_{\mathcal{M}_n}\boldsymbol{u}_n\,\mathrm{div}(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x} + \mu\int_\Omega \mathrm{curl}_{\mathcal{M}_n}\boldsymbol{u}_n\cdot\mathrm{curl}(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x} + R_3^n \quad (5.32)$$

with R_3^n which has the following structure:

$$R_3^n = \mu \sum_{\sigma\in\mathcal{E}_{n,\mathrm{int}}} \int_\sigma \sum_{1\leq i,j,k\leq 3} n_{\sigma,i,j,k}\,[(\boldsymbol{u}_n)_i]_\sigma\,(\boldsymbol{\nabla}(\phi\boldsymbol{w}_n))_{j,k}\ \mathrm{d}\sigma(\boldsymbol{x}), \quad (5.33)$$

where for $\sigma\in\mathcal{E}_{\mathrm{int}}$ and $1\leq i,j,k\leq 3$, $n_{\sigma,i,j,k}$ is a component of the unit normal vector to σ. Injecting (5.32) in (5.31) we get:

$$- \int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x}$$
$$+ (2\mu+\lambda)\int_\Omega \mathrm{div}_{\mathcal{M}_n}\boldsymbol{u}_n\,\mathrm{div}(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x} + \mu\int_\Omega \mathrm{curl}_{\mathcal{M}_n}\boldsymbol{u}_n\cdot\mathrm{curl}(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x}$$
$$- a\int_\Omega \rho_n^\gamma\,\mathrm{div}(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x} + R_3^n + \underset{n\to+\infty}{o}(1) = \int_\Omega \boldsymbol{f}\cdot(\phi\boldsymbol{w}_n)\ \mathrm{d}\boldsymbol{x}. \quad (5.34)$$

By Lemma 5.9 with $q=2$, we have:

$$|R_3^n| \lesssim h_n\,\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n}\,|\boldsymbol{\nabla}(\phi\boldsymbol{w}_n)|_{\mathbf{H}^1(\Omega)} \lesssim h_n\,\|\boldsymbol{w}_n\|_{\mathbf{W}^{2,\frac{1+\eta}{\eta}}(\Omega)} \lesssim h_n^{1-\xi}.$$

The choice of $\boldsymbol{w}_n$ gives $\mathrm{div}(\phi \boldsymbol{w}_n) = i_{\mathcal{M}_n} T_k(\rho_n)\,\phi + \boldsymbol{w}_n \cdot \boldsymbol{\nabla}\phi$ and $\mathrm{curl}(\phi \boldsymbol{w}_n) = L(\phi)\,\boldsymbol{w}_n$, where $L(\phi)$ is a matrix with entries involving first order derivatives of ϕ. Hence, reordering (5.34) we have:

$$\int_\Omega \big((2\mu+\lambda)\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n - a\rho_n^\gamma\big)\, T_k(\rho_n)\,\phi \,\mathrm{d}\boldsymbol{x} + R_4^n + \underset{n\to+\infty}{o}(1)$$

$$= \int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) \otimes (\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \,\mathrm{d}\boldsymbol{x} - (2\mu+\lambda)\int_\Omega \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n\,(\boldsymbol{w}_n\cdot\boldsymbol{\nabla}\phi)\,\mathrm{d}\boldsymbol{x}$$

$$- \mu \int_\Omega \mathrm{curl}_{\mathcal{M}_n}\boldsymbol{u}_n \cdot (L(\phi)\,\boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + a\int_\Omega \rho_n^\gamma\,\boldsymbol{w}_n\cdot\boldsymbol{\nabla}\phi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}\cdot(\phi\boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x}, \quad (5.35)$$

with

$$R_4^n = \int_\Omega \big((2\mu+\lambda)\,\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n - a\rho_n^\gamma\big)\big(i_{\mathcal{M}_n}T_k(\rho_n) - T_k(\rho_n)\big)\phi\,\mathrm{d}\boldsymbol{x}.$$

Since $(\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$ and $(\rho_n^\gamma)_{n\in\mathbb{N}}$ in $\mathrm{L}^{1+\eta}(\Omega)$, estimate (5.27) (with $\frac{1+\eta}{\eta}\geq 2$) yields $R_4^n \to 0$ as $n\to+\infty$. Moreover, we know that $\mathrm{div}_{\mathcal{M}_n}\boldsymbol{u}_n$, $\mathrm{curl}_{\mathcal{M}_n}\boldsymbol{u}_n$ (*resp.* ρ_n^γ) weakly converge in $\mathrm{L}^2(\Omega)$ (*resp.* in $\mathrm{L}^{1+\eta}(\Omega)$) respectively towards $\mathrm{div}\,\boldsymbol{u}$, $\mathrm{curl}\,\boldsymbol{u}$ and $\overline{\rho^\gamma}$. Since $\boldsymbol{w}_n$ strongly converges in $\mathbf{L}^q(\Omega)$ towards $\boldsymbol{w}$ for all $q\in(1,+\infty)$, we get, passing to the limit $n\to+\infty$ in (5.35):

$$\lim_{n\to+\infty} \int_\Omega \big((2\mu+\lambda)\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n - a\rho_n^\gamma\big)\, T_k(\rho_n)\,\phi\,\mathrm{d}\boldsymbol{x}$$

$$= \lim_{n\to+\infty}\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n\,\mathrm{d}\boldsymbol{x}$$

$$- (2\mu+\lambda)\int_\Omega \mathrm{div}\,\boldsymbol{u}\,(\boldsymbol{w}\cdot\boldsymbol{\nabla}\phi)\,\mathrm{d}\boldsymbol{x} - \mu\int_\Omega \mathrm{curl}\,\boldsymbol{u}\cdot(L(\phi)\,\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}$$

$$+ a\int_\Omega \overline{\rho^\gamma}\,\boldsymbol{w}\cdot\boldsymbol{\nabla}\phi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}\cdot(\phi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}. \quad (5.36)$$

Let us now determine the limit of the convective term in the right hand side of (5.36). As in the continuous case, we introduce a mollifying sequence $(\boldsymbol{\omega}_\delta)_{\delta>0}$ and the regularized velocities $\boldsymbol{u}_{n,\delta} = \boldsymbol{u}_n * \boldsymbol{\omega}_\delta$ and $\boldsymbol{u}_\delta = \boldsymbol{u} * \boldsymbol{\omega}_\delta$ where $\boldsymbol{u}_n$ and $\boldsymbol{u}$ have been extended by 0 outside Ω. We have $\boldsymbol{u}_{n,\delta}\in\mathbf{L}^6(\Omega)$ with $\|\boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^6(\Omega)} \leq C\|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)}$ and for $q\in(6,+\infty]$, $\boldsymbol{u}_{n,\delta}\in\mathbf{L}^q(\Omega)$ with $\|\boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\Omega)} \leq C_\delta\|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)}$. Moreover, for all $m\in\mathbb{N}$ and $q\in[1,+\infty]$, $\boldsymbol{u}_{n,\delta}\in\mathbf{W}^{m,q}(\Omega)$ with $|\boldsymbol{u}_{n,\delta}|_{\mathbf{W}^{m,q}} \leq C_\delta\|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)}$. Furthermore, we recall that

$$\boldsymbol{u}_{n,\delta} \underset{n\to+\infty}{\longrightarrow} \boldsymbol{u}_\delta \quad \text{strongly in } \mathbf{L}^q_{\mathrm{loc}}(\mathbb{R}^3)\ \forall q\in[1,6) \text{ uniformly in } \delta, \quad (5.37)$$

$$\boldsymbol{u}_{n,\delta} \underset{\delta\to 0}{\longrightarrow} \boldsymbol{u}_n \quad \text{strongly in } \mathbf{L}^q_{\mathrm{loc}}(\mathbb{R}^3)\ \forall q\in[1,6) \text{ (uniformly in } n), \quad (5.38)$$

$$\boldsymbol{u}_\delta \underset{\delta\to 0}{\longrightarrow} \boldsymbol{u} \quad \text{strongly in } \mathbf{L}^6_{\mathrm{loc}}(\mathbb{R}^3). \quad (5.39)$$

Denoting $\tilde{\boldsymbol{u}}_{n,\delta} = I_{\mathcal{M}_n} \boldsymbol{u}_{n,\delta}$, we have:

$$
\begin{aligned}
-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} \\
= -\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta})\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} + R_5^{n,\delta}
\end{aligned}
\tag{5.40}
$$

with

$$
R_5^{n,\delta} = -\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,\big(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n - (\Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta})\big)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x}.
$$

Since $(\rho_n\boldsymbol{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^p(\Omega)$ for some $p > \frac{6}{5}$, $(\boldsymbol{\nabla}\boldsymbol{w}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^s(\Omega)^3$ for any $s \in (1,+\infty)$, then the following inequality holds, for some triple (p,q,s), such that $p > \frac{6}{5}$, $s > 1$, $q < 6$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$:

$$
\begin{aligned}
|R_5^{n,\delta}| &\lesssim \|\rho_n\boldsymbol{u}_n\|_{\mathrm{L}^p(\Omega)} \|\boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n\|_{\mathbf{L}^s(\Omega)^3} \|\Pi_{\mathcal{E}_n}\boldsymbol{u}_n - \Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^q(\Omega)} \\
&\lesssim \|\rho_n\boldsymbol{u}_n\|_{\mathrm{L}^p(\Omega)} \|\boldsymbol{\nabla}\boldsymbol{w}_n\|_{\mathbf{L}^s(\Omega)^3} \|\Pi_{\mathcal{E}_n}\boldsymbol{u}_n - \Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^q(\Omega)} \\
&\lesssim \|\Pi_{\mathcal{E}_n}\boldsymbol{u}_n - \Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^q(\Omega)} \\
&\lesssim \|\boldsymbol{u}_n - \tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^q(\Omega)} \\
&\lesssim \big(\|\boldsymbol{u}_n - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\Omega)} + \|\tilde{\boldsymbol{u}}_{n,\delta} - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\Omega)}\big)
\end{aligned}
$$

where the constants involved in these inequalities are independent of n and δ. From Lemma 5.5 we have:

$$
\|\tilde{\boldsymbol{u}}_{n,\delta} - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\Omega)} \lesssim h_n^2 |\boldsymbol{u}_{n,\delta}|_{\mathbf{W}^{2,q}(\Omega)} \lesssim C_\delta h_n^2 \|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)} \lesssim C_\delta h_n^2.
$$

Therefore:

$$
\limsup_{n\to+\infty} |R_5^{n,\delta}| \lesssim \limsup_{n\to+\infty} \|\boldsymbol{u}_n - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\Omega)}, \tag{5.41}
$$

where the involved constant is independent of n and δ. Let us now deal with the integral in the right hand side of (5.40). Performing a discrete integration by parts we get:

$$
\begin{aligned}
-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta})\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} \\
= -\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}} \\ \sigma=K|L}} |\sigma|\,\rho_\sigma\,(\tilde{\boldsymbol{u}}_{n,\delta})_\sigma \otimes \boldsymbol{u}_\sigma : (\boldsymbol{v}_L - \boldsymbol{v}_K)\otimes\boldsymbol{n}_{K,\sigma} \\
= -\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}} \\ \sigma=K|L}} |\sigma|\,\rho_\sigma\,(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\,((\tilde{\boldsymbol{u}}_{n,\delta})_\sigma\cdot(\boldsymbol{v}_L - \boldsymbol{v}_K)) \\
= \sum_{K\in\mathcal{M}_n} \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\rho_\sigma\,(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\,(\tilde{\boldsymbol{u}}_{n,\delta})_\sigma\Big)\cdot\boldsymbol{v}_K.
\end{aligned}
$$

Injecting $(\tilde{\boldsymbol{u}}_{n,\delta})_\sigma = (\tilde{\boldsymbol{u}}_{n,\delta})_\sigma - (\tilde{\boldsymbol{u}}_{n,\delta})_K + (\tilde{\boldsymbol{u}}_{n,\delta})_K$, where $(\tilde{\boldsymbol{u}}_{n,\delta})_K$ is the mean value of the function $\tilde{\boldsymbol{u}}_{n,\delta}$ over K, we get:

$$
\begin{aligned}
-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta})\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n \ \mathrm{d}\boldsymbol{x} \\
= \sum_{K\in\mathcal{M}_n} \Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\rho_\sigma\,(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\,\big((\tilde{\boldsymbol{u}}_{n,\delta})_\sigma - (\tilde{\boldsymbol{u}}_{n,\delta})_K\big)\Big)\cdot\boldsymbol{v}_K + R_6^{n,\delta} + R_7^{n,\delta}
\end{aligned}
\tag{5.42}
$$

where, using the discrete mass conservation equation (3.4a), we have:

$$R_6^{n,\delta} = -h_n^{\xi_1} \sum_{K\in\mathcal{M}_n} |K|(\rho_K - \rho^\star)\,(\tilde{\boldsymbol{u}}_{n,\delta})_K \cdot \boldsymbol{v}_K,$$

$$R_7^{n,\delta} = h_n^{\xi_2} \sum_{K\in\mathcal{M}_n} \Big(\sum_{\substack{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}-1} (\rho_K-\rho_L)\Big) (\tilde{\boldsymbol{u}}_{n,\delta})_K \cdot \boldsymbol{v}_K.$$

Since $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{\frac{3}{2}}(\Omega)$, $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^\infty(\Omega)$ and

$$\|\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^6(\Omega)} \lesssim \|\boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^6(\Omega)} \lesssim 1$$

where the involved constants are independent of n and δ, we obtain that:

$$|R_6^{n,\delta}| \to 0 \quad \text{ as } n\to+\infty \text{ uniformly with respect to } \delta>0. \tag{5.43}$$

Reordering the sum in $R_7^{n,\delta}$ we get:

$$\begin{aligned} R_7^{n,\delta} &= -h_n^{\xi_2} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}-1} (\rho_K-\rho_L) \big((\tilde{\boldsymbol{u}}_{n,\delta})_L\cdot\boldsymbol{v}_L - (\tilde{\boldsymbol{u}}_{n,\delta})_K\cdot\boldsymbol{v}_K\big) \\ &= R_{7,1}^{n,\delta} + R_{7,2}^{n,\delta}, \end{aligned}$$

where

$$R_{7,1}^{n,\delta} = -h_n^{\xi_2} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}-1} (\rho_K-\rho_L)\, \boldsymbol{v}_L \cdot \big((\tilde{\boldsymbol{u}}_{n,\delta})_L - (\tilde{\boldsymbol{u}}_{n,\delta})_K\big),$$

$$R_{7,2}^{n,\delta} = -h_n^{\xi_2} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}-1} (\rho_K-\rho_L)\, (\tilde{\boldsymbol{u}}_{n,\delta})_K \cdot (\boldsymbol{v}_L - \boldsymbol{v}_K).$$

The first term is controlled as follows:

$$\begin{aligned} |R_{7,1}^{n,\delta}| &\le h_n^{\xi_2} \|\boldsymbol{v}_n\|_{\mathbf{L}^\infty(\Omega)} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |D_\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\,|\rho_K-\rho_L|\Big)^{\frac{1}{\eta}} \frac{|\sigma|}{|D_\sigma|} \big|(\tilde{\boldsymbol{u}}_{n,\delta})_L - (\tilde{\boldsymbol{u}}_{n,\delta})_K\big| \\ &\lesssim h_n^{\xi_2} \, \||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^{1+\eta}(\Omega)} \Big(\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |D_\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1+\eta}{\eta}} \big|(\tilde{\boldsymbol{u}}_{n,\delta})_L - (\tilde{\boldsymbol{u}}_{n,\delta})_K\big|^{\frac{1+\eta}{\eta}} \Big)^{\frac{\eta}{1+\eta}} \end{aligned}$$

where, following similar steps as in the proof of Proposition 5.2 (see the calculation after eq. (5.12)), we have:

$$\begin{aligned} \big|(\tilde{\boldsymbol{u}}_{n,\delta})_L - (\tilde{\boldsymbol{u}}_{n,\delta})_K\big|^{\frac{1+\eta}{\eta}} &\lesssim \big|(\tilde{\boldsymbol{u}}_{n,\delta})_L - (\tilde{\boldsymbol{u}}_{n,\delta})_\sigma\big|^{\frac{1+\eta}{\eta}} + \big|(\tilde{\boldsymbol{u}}_{n,\delta})_K - (\tilde{\boldsymbol{u}}_{n,\delta})_\sigma\big|^{\frac{1+\eta}{\eta}} \\ &\lesssim \frac{h_L^{\frac{1+\eta}{\eta}}}{|L|} \|\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_{n,\delta}\|^{\frac{1+\eta}{\eta}}_{\mathbf{L}^{\frac{1+\eta}{\eta}}(L)^3} + \frac{h_K^{\frac{1+\eta}{\eta}}}{|K|} \|\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_{n,\delta}\|^{\frac{1+\eta}{\eta}}_{\mathbf{L}^{\frac{1+\eta}{\eta}}(K)^3}. \end{aligned}$$

By the regularity of the sequence of discretizations, we get:

$$\begin{aligned}|R_{7,1}^{n,\delta}| &\lesssim h_n^{\xi_2}\, \||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^{1+\eta}(\Omega)}\, \|\tilde{\boldsymbol{u}}_{n,\delta}\|_{1,\frac{1+\eta}{\eta},\mathcal{M}_n} \\ &\lesssim h_n^{\xi_2}\, \||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^{1+\eta}(\Omega)}\, |\boldsymbol{u}_{n,\delta}|_{\mathbf{W}^{1,\frac{1+\eta}{\eta}}(\Omega)} \\ &\lesssim C_\delta h_n^{\xi_2}\, \||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^{1+\eta}(\Omega)}\, \|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)} \\ &\lesssim C_\delta h_n^{\xi_2}\, \||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^{1+\eta}(\Omega)},\end{aligned}$$

where the constants involved in $\lesssim$ are independent of n (and δ). By the uniform estimate (5.4) we have

$$\||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^{1+\eta}(\Omega)} = \|\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)\|^{\frac{1}{\eta}}_{\mathbf{L}^{\frac{1+\eta}{\eta}}(\Omega)} \lesssim h_n^{-\frac{1}{1+\eta}(\xi_2+\frac{5}{4\Gamma}(\frac{3}{1+\eta}+\xi_3))}$$

Therefore

$$|R_{7,1}^{n,\delta}| \lesssim C_\delta h_n^{\frac{\eta}{1+\eta}(\xi_2-\frac{5}{4\eta\Gamma}(\frac{3}{1+\eta}+\xi_3))}$$

which yields, with condition (3.21):

$$|R_{7,1}^{n,\delta}| \to 0 \quad \text{as } n \to +\infty \text{ for any fixed } \delta > 0. \tag{5.44}$$

The second term is controlled in a similar way:

$$\begin{aligned}|R_{7,2}^{n,\delta}| &\le h_n^{\xi_2} \|\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^\infty(\Omega)} \sum_{\substack{\sigma\in\mathcal{E}_{n,\text{int}}\\ \sigma=K|L}} |D_\sigma| \left(\frac{|\sigma|}{D_\sigma}\,|\rho_K-\rho_L|\right)^{\frac{1}{\eta}} \frac{|\sigma|}{|D_\sigma|}(\boldsymbol{v}_L-\boldsymbol{v}_K) \\ &\lesssim h_n^{\xi_2} \|\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^\infty(\Omega)}\, \||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^{1+\eta}(\Omega)}\, \|\boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n\|_{\mathbf{L}^{\frac{\eta}{1+\eta}}(\Omega)} \\ &\lesssim h_n^{\xi_2} \|\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^\infty(\Omega)}\, \||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^{1+\eta}(\Omega)}\, \|\boldsymbol{v}_n\|_{1,\frac{\eta}{1+\eta},\mathcal{M}_n} \\ &\lesssim h_n^{\xi_2} \|\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^\infty(\Omega)}\, \|\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)\|^{\frac{1}{\eta}}_{\mathbf{L}^{\frac{1+\eta}{\eta}}(\Omega)}\, |\boldsymbol{w}_n|_{\mathbf{W}^{1,\frac{\eta}{1+\eta}}(\Omega)} \\ &\lesssim h_n^{\frac{\eta}{1+\eta}(\xi_2-\frac{5}{4\eta\Gamma}(\frac{3}{1+\eta}+\xi_3))} \|\tilde{\boldsymbol{u}}_{n,\delta}\|_{\mathbf{L}^\infty(\Omega)}, \\ &\lesssim C_\delta h_n^{\frac{\eta}{1+\eta}(\xi_2-\frac{5}{4\eta\Gamma}(\frac{3}{1+\eta}+\xi_3))},\end{aligned}$$

where the involved constants are independent of n (and δ). Using again (3.21), this implies that

$$|R_{7,2}^{n,\delta}| \to 0 \quad \text{as } n \to +\infty \text{ for any fixed } \delta > 0. \tag{5.45}$$

Let $\boldsymbol{Q}_{n,\delta}$ and $\Pi_{\mathcal{M}_n}\boldsymbol{v}_n$ be the functions defined by:

$$\boldsymbol{Q}_{n,\delta}(\boldsymbol{x}) = \sum_{K\in\mathcal{M}_n} \frac{1}{|K|}\Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\rho_\sigma\,(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\,\big((\tilde{\boldsymbol{u}}_{n,\delta})_\sigma-(\tilde{\boldsymbol{u}}_{n,\delta})_K\big)\Big)\mathcal{X}_K(\boldsymbol{x}),$$

$$\Pi_{\mathcal{M}_n}\boldsymbol{v}_n(\boldsymbol{x}) = \sum_{K\in\mathcal{M}_n} \boldsymbol{v}_K\,\mathcal{X}_K(\boldsymbol{x}),$$

so that, back to (5.42), we have :

$$-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta})\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n):\boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{Q}_{n,\delta}\cdot\Pi_{\mathcal{M}_n}\boldsymbol{v}_n\ \mathrm{d}\boldsymbol{x} + R_6^{n,\delta} + R_{7,1}^{n,\delta} + R_{7,2}^{n,\delta}. \quad (5.46)$$

Let us prove that, for a fixed $\delta > 0$, $\boldsymbol{Q}_{n,\delta}$ weakly converges (up to a subsequence) in $\mathbf{L}^r(\Omega)$ for some $r > 1$ towards $\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_\delta$ as $n \to +\infty$. The sum in $\boldsymbol{Q}_{n,\delta}(\boldsymbol{x})$ can be rearranged as follows:

$$\boldsymbol{Q}_{n,\delta}(\boldsymbol{x}) = \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} \rho_\sigma \left(\frac{|\sigma|}{|K|}\big((\tilde{\boldsymbol{u}}_{n,\delta})_\sigma - (\tilde{\boldsymbol{u}}_{n,\delta})_K\big)\,(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\mathcal{X}_K(\boldsymbol{x}) + \frac{|\sigma|}{|L|}\big((\tilde{\boldsymbol{u}}_{n,\delta})_\sigma - (\tilde{\boldsymbol{u}}_{n,\delta})_L\big)\,(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{L,\sigma})\mathcal{X}_L(\boldsymbol{x})\right).$$

Proceeding as above for the control of $\big|(\tilde{\boldsymbol{u}}_{n,\delta})_K - (\tilde{\boldsymbol{u}}_{n,\delta})_\sigma\big|^6$, and invoking once again the following estimates

$$\|\tilde{\boldsymbol{u}}_{n,\delta}\|_{1,6,\mathcal{M}_n} \lesssim |\boldsymbol{u}_{n,\delta}|_{\mathbf{W}^{1,6}(\Omega)} \lesssim C_\delta\|\boldsymbol{u}_n\|_{\mathbf{L}^6(\Omega)} \le C_\delta\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n},$$

combined with the estimates on $\mathcal{P}_{\mathcal{E}_n}\rho_n$ in $\mathrm{L}^{3(\gamma-1)}(\Omega)$, $\Pi_{\mathcal{E}_n}\boldsymbol{u}_n$ in $\mathbf{L}^6(\Omega)$, we can prove that $(\boldsymbol{Q}_{n,\delta})_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^r(\Omega)$ with $r > 1$ (because $3(\gamma-1) > \frac{3}{2}$). Then, up to a subsequence, $\boldsymbol{Q}_{n,\delta}$ weakly converges towards some $\boldsymbol{Q}_\delta$ in $\mathbf{L}^r(\Omega)$ as $n \to +\infty$.

Let us now identify $\boldsymbol{Q}_\delta$. Let $\boldsymbol{\psi} \in \mathcal{C}_c^\infty(\Omega)^3$ and denote $\boldsymbol{\psi}_n = I_{\mathcal{M}_n}\boldsymbol{\psi}$. Since $\boldsymbol{\psi}$ is smooth, we have $\Pi_{\mathcal{M}_n}\boldsymbol{\psi}_n \to \boldsymbol{\psi}$ in $\mathbf{L}^{r'}(\Omega)$ (with $\frac{1}{r}+\frac{1}{r'}=1$). Hence we have (observing that $R_6^{n,\delta} \to 0$ and $R_7^{n,\delta} \to 0$ as $n \to +\infty$ with $\boldsymbol{\psi}_n$ instead of $\boldsymbol{v}_n$):

$$\int_\Omega \boldsymbol{Q}_\delta\cdot\boldsymbol{\psi}\ \mathrm{d}\boldsymbol{x} = \lim_{n\to+\infty}\int_\Omega \boldsymbol{Q}_{n,\delta}\cdot\Pi_{\mathcal{M}_n}\boldsymbol{\psi}_n\ \mathrm{d}\boldsymbol{x} = \lim_{n\to+\infty} -\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\tilde{\boldsymbol{u}}_{n,\delta})\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n):\boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{\psi}_n\ \mathrm{d}\boldsymbol{x}.$$

Since $\tilde{\boldsymbol{u}}_{n,\delta}$ converges strongly as $n \to +\infty$ to $\boldsymbol{u}_\delta$ in $\mathbf{L}^q(\Omega)$ for all $q < 6$ (uniformly with respect to δ) and $\|\tilde{\boldsymbol{u}}_{n,\delta}\|_{1,2,\mathcal{M}_n} \le C_\delta\|\boldsymbol{u}_n\|_{1,2,\mathcal{M}_n}$, we can reproduce the same arguments as those used in the previous Subsection 5.2 (passing to the limit in the momentum equation) and obtain:

$$\int_\Omega \boldsymbol{Q}_\delta\cdot\boldsymbol{\psi}\ \mathrm{d}\boldsymbol{x} = -\int_\Omega \rho\boldsymbol{u}_\delta\otimes\boldsymbol{u}:\boldsymbol{\nabla}\boldsymbol{\psi}\ \mathrm{d}\boldsymbol{x} = -\int_\Omega \boldsymbol{u}_\delta\otimes(\rho\boldsymbol{u}):\boldsymbol{\nabla}\boldsymbol{\psi}\ \mathrm{d}\boldsymbol{x}.$$

Since the limit functions satisfy $(\rho,\boldsymbol{u}) \in \mathrm{L}^{3(\gamma-1)}(\Omega)\times\mathbf{H}_0^1(\Omega)$ and since we have already proved that $\mathrm{div}(\rho\boldsymbol{u}) = 0$ in Section 5.1, we infer that:

$$\boldsymbol{Q}_\delta = \rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}_\delta.$$

Back to (5.40) and (5.46) we get :

$$-\int_\Omega (\mathcal{P}_{\mathcal{E}_n}\rho_n)\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\otimes(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n):\boldsymbol{\nabla}_{\mathcal{E}_n}\boldsymbol{v}_n\ \mathrm{d}\boldsymbol{x} + \int_\Omega \rho\boldsymbol{u}\otimes\boldsymbol{u}:\boldsymbol{\nabla}(\phi\boldsymbol{w})\ \mathrm{d}\boldsymbol{x} = R_5^{n,\delta} + R_6^{n,\delta} + R_{7,1}^{n,\delta} + R_{7,2}^{n,\delta} + R_8^{n,\delta} + R_9^{\delta}. \quad (5.47)$$

where

$$R_8^{n,\delta} = \int_\Omega \boldsymbol{Q}_{n,\delta} \cdot \Pi_{\mathcal{M}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{Q}_k \cdot (\phi \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x},$$

$$R_9^{\delta} = \int_\Omega \rho(\boldsymbol{u} - \boldsymbol{u}_\delta) \otimes \boldsymbol{u} : \boldsymbol{\nabla}(\phi \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x}.$$

The function $\Pi_{\mathcal{M}_n} \boldsymbol{v}_n$ converges to $\phi \boldsymbol{w}$ strongly in $\mathbf{L}^{r'}(\Omega)$ as $n \to +\infty$. Indeed, we know that in $\mathbf{L}^{r'}(\Omega)$, $\phi \boldsymbol{w}_n \to \phi \boldsymbol{w}$ and $\boldsymbol{\delta}_n = \boldsymbol{v}_n - \phi \boldsymbol{w}_n \to 0$ as $n \to +\infty$ and we also have $\Pi_{\mathcal{M}_n} \boldsymbol{v}_n - \boldsymbol{v}_n \to 0$ in $\mathbf{L}^{r'}(\Omega)$ since:

$$\begin{aligned}
\|\Pi_{\mathcal{M}_n} \boldsymbol{v}_n - \boldsymbol{v}_n\|_{\mathbf{L}^{r'}(\Omega)}^{r'} &= \sum_{K \in \mathcal{M}_n} \int_K \Big| \sum_{\sigma, \sigma' \in \mathcal{E}(K)} (\boldsymbol{v}_\sigma - \boldsymbol{v}_{\sigma'}) \xi_K^\sigma \, \zeta_{\sigma'}(\boldsymbol{x}) \Big|^{r'} \, \mathrm{d}\boldsymbol{x} \\
&\lesssim h_n^{r'} \sum_{K \in \mathcal{M}_n} h_K^{3-r'} \sum_{\sigma, \sigma' \in \mathcal{E}(K)} |\boldsymbol{v}_\sigma - \boldsymbol{v}_{\sigma'}|^{r'}.
\end{aligned}$$

Hence we have $\|\Pi_{\mathcal{M}_n} \boldsymbol{v}_n - \boldsymbol{v}_n\|_{\mathbf{L}^{r'}(\Omega)} \lesssim h_n \, \|\boldsymbol{v}_n\|_{1,r',\mathcal{E}_n} \lesssim h_n \, \|\boldsymbol{v}_n\|_{1,r',\mathcal{M}_n} \lesssim h_n |\phi \boldsymbol{w}_n|_{\mathbf{W}^{1,r'}(\Omega)} \lesssim h_n$. Therefore, by the weak convergence of $\boldsymbol{Q}_{n,\delta}$ towards $\boldsymbol{Q}_\delta$ in $\mathbf{L}^r(\Omega)$ we have:

$$|R_8^{n,\delta}| \to 0 \quad \text{as } n \to +\infty \text{ for any fixed } \delta > 0. \tag{5.48}$$

Combining the estimates (5.41)-(5.43)-(5.44)-(5.45)-(5.48) and passing to limit $n \to +\infty$ in (5.47), we obtain that:

$$\begin{aligned}
\limsup_{n \to +\infty} \Big| \int_\Omega (\mathcal{P}_{\mathcal{E}_n} \rho_n) \, (\Pi_{\mathcal{E}_n} \boldsymbol{u}_n) \otimes (\Pi_{\mathcal{E}_n} \boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla}(\phi \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x} \Big| \\
\lesssim \limsup_{n \to +\infty} \|\boldsymbol{u}_n - \boldsymbol{u}_{n,\delta}\|_{\mathbf{L}^q(\Omega)} + |R_9^\delta|, \quad (5.49)
\end{aligned}$$

for some $q \in [1, 6)$ and for all $\delta > 0$. By (5.39) we have $R_9^\delta \to 0$ as $\delta \to 0$ and by the uniform in n convergence (5.38) we finally obtain, letting $\delta \to 0$ in (5.49) that:

$$\lim_{n \to +\infty} \int_\Omega (\mathcal{P}_{\mathcal{E}_n} \rho_n) \, (\Pi_{\mathcal{E}_n} \boldsymbol{u}_n) \otimes (\Pi_{\mathcal{E}_n} \boldsymbol{u}_n) : \boldsymbol{\nabla}_{\mathcal{E}_n} \boldsymbol{v}_n \, \mathrm{d}\boldsymbol{x} = \int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla}(\phi \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x}.$$

Going back to (5.36) we obtain:

$$\begin{aligned}
\lim_{n \to +\infty} \int_\Omega \big((2\mu + \lambda) \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n - a \rho_n^\gamma\big) \, T_k(\rho_n) \, \phi \, \mathrm{d}\boldsymbol{x} \\
= \int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\nabla}(\phi \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x} - (2\mu + \lambda) \int_\Omega \mathrm{div} \, \boldsymbol{u} \, (\boldsymbol{w} \cdot \boldsymbol{\nabla} \phi) \, \mathrm{d}\boldsymbol{x} \\
- \mu \int_\Omega \mathrm{curl} \, \boldsymbol{u} \cdot (L(\phi) \, \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x} + a \int_\Omega \overline{\rho^\gamma} \, \boldsymbol{w} \cdot \boldsymbol{\nabla} \phi \, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f} \cdot (\phi \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x}. \quad (5.50)
\end{aligned}$$

Applying the identity (2.19) to the functions $\boldsymbol{u}$ and $\phi \boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$, we get:

$$\begin{aligned}\lim_{n\to+\infty} &\int_\Omega \left((2\mu+\lambda)\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n - a\rho_n^\gamma\right) T_k(\rho_n)\, \phi \,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \left((2\mu+\lambda)\,\mathrm{div}\, \boldsymbol{u} - a\overline{\rho^\gamma}\right) \overline{T_k(\rho)}\, \phi \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho\, \boldsymbol{u}\otimes \boldsymbol{u} : \boldsymbol{\nabla}(\phi \boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} \\ &\quad - \mu \int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}(\phi\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} - (\mu+\lambda)\int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}(\phi\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} \\ &\quad + a\int_\Omega \overline{\rho^\gamma}\,\mathrm{div}(\phi\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}\cdot(\phi\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

We have already proved that the limit triple $(\rho, \boldsymbol{u}, \overline{\rho^\gamma}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathrm{L}^{\frac{3(\gamma-1)}{\gamma}}$ satisfies the momentum equation in the weak sense. Thus, applying Proposition 5.4 to $\boldsymbol{v} = \phi\boldsymbol{w}$ (using the density of $\mathcal{C}_c^\infty(\Omega)^3$ in $\mathbf{W}_0^{1,q}(\Omega)$ for all $q \in [1, +\infty)$) yields

$$\lim_{n\to+\infty} \int_\Omega \left((2\mu+\lambda)\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n - a\rho_n^\gamma\right) T_k(\rho_n)\, \phi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \left((2\mu+\lambda)\,\mathrm{div}\, \boldsymbol{u} - a\overline{\rho^\gamma}\right) \overline{T_k(\rho)}\, \phi \,\mathrm{d}\boldsymbol{x},$$

thus concluding the proof of Lemma 5.7. □

5.3.2 Strong convergence of the density and renormalization property

Properties of the truncation operators T_k. We first state two results that are the discrete counterparts of Lemmas 2.10-2.11.

Lemma 5.10. *There exists a constant C such that the following inequality holds for all $1 \le q < 3(\gamma-1)$, $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$:*

$$\|\overline{T_k(\rho)} - \rho\|_{\mathrm{L}^q(\Omega)} + \|T_k(\rho) - \rho\|_{\mathrm{L}^q(\Omega)} + \|T_k(\rho_n) - \rho_n\|_{\mathrm{L}^q(\Omega)} \le C k^{\frac{1}{3(\gamma-1)} - \frac{1}{q}}.$$

Consequently, as $k \to +\infty$, the sequences $(\overline{T_k(\rho)})_{k\in\mathbb{N}^*}$ and $(T_k(\rho))_{k\in\mathbb{N}^*}$ both converge strongly to ρ in $\mathrm{L}^q(\Omega)$ for all $q \in [1, 3(\gamma-1))$.

Proof. The proof is similar to that of Lemma 2.10 in the continuous case. It relies on the fact that $\int_\Omega \rho_n = |\Omega|\rho^\star$ and on the uniform bound on $(\|\rho_n\|_{\mathrm{L}^{3(\gamma-1)}(\Omega)})_{n\in\mathbb{N}}$. □

Lemma 5.11. *There exists a constant C such that the following estimate holds:*

$$\sup_{k>1} \limsup_{n\to+\infty} \|T_k(\rho_n) - T_k(\rho)\|_{\mathrm{L}^{\gamma+1}(\Omega)} \le C. \tag{5.51}$$

Proof. Here again, the proof is similar to that of Lemma 2.11 in the continuous case. It relies on the convergence of the effective viscous flux obtained in Proposition 5.7 and on the uniform bound on $(\|\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n\|_{\mathrm{L}^2(\Omega)})_{n\in\mathbb{N}}$. □

Renormalization equation associated with T_k. We first state a discrete renormalization property for truncated functions which is an analogous of the renormalization property stated in Remark 2.4. The proof is similar to that of Prop. 4.8 which is given in Appendix A.1.

Proposition 5.12. *For any $b \in \mathcal{C}^1([0,+\infty))$, denote b_M the truncated function such that*

$$b_M(t) = \begin{cases} b(t) & \text{if } t < M, \\ b(M) & \text{if } t \geq M, \end{cases}$$

and $[b_M]'_+$ its discontinuous derivative:

$$[b_M]'_+(t) = \begin{cases} b'(t) & \text{if } t < M, \\ 0 & \text{if } t \geq M. \end{cases}$$

Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. If $(\rho, \boldsymbol{u}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M},0}(\Omega)$ satisfy the discrete mass balance (3.4a) with $\rho > 0$ a.e. in Ω (i.e. $\rho_K > 0$, $\forall K \in \mathcal{M}$) then we have:

$$\operatorname{div}\big(b_M(\rho)\boldsymbol{u}\big)_K + \big([b_M]'_+(\rho_K)\rho_K - b_M(\rho_K)\big)\operatorname{div}(\boldsymbol{u})_K + R^1_K + R^2_K + R^3_K = 0 \quad \forall K \in \mathcal{M}, \tag{5.52}$$

where

$$\operatorname{div}\big(b_M(\rho)\boldsymbol{u}\big)_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \; b_M(\rho_\sigma) \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma},$$

and

$$R^1_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| r_{K,\sigma} \left(\boldsymbol{u}_\sigma \cdot n_{K,\sigma}\right) \quad \text{and} \quad r_{K,\sigma} = [b_M]'_+(\rho_K)(\rho_\sigma - \rho_K) + b_M(\rho_K) - b_M(\rho_\sigma),$$

$$R^2_K = h_{\mathcal{M}}^{\xi_2} \; [b_M]'_+(\rho_K) \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta} - 1} (\rho_K - \rho_L),$$

$$R^3_K = h_{\mathcal{M}}^{\xi_1} [b_M]'_+(\rho_K)(\rho_K - \rho^\star).$$

Now, for any $k \in \mathbb{N}^*$ we consider the function L_k introduced in Section 2 and defined as

$$L_k(t) = \begin{cases} t(\ln t - \ln k - 1), & \text{if } t \in [0,k), \\ -k, & \text{if } t \in [k, +\infty). \end{cases}$$

We recall that $L_k \in \mathcal{C}^0([0,+\infty)) \cap \mathcal{C}^1((0,+\infty))$ and

$$tL'_k(t) - L_k(t) = T_k(t) \quad \forall t \in [0, +\infty).$$

Proposition 5.13. *Under the assumptions of Theorem 3.1, let $(\rho, \boldsymbol{u}) \in \mathrm{L}^{3(\gamma-1)}(\Omega) \times \mathbf{H}^1_0(\Omega)$ be the limit couple of the sequence $(\rho_n, \boldsymbol{u}_n)_{n \in \mathbb{N}}$. Then, for all $k \in \mathbb{N}^*$, the following inequalities hold:*

$$\operatorname{div}_{\mathcal{M}_n}(L_k(\rho_n)\boldsymbol{u}_n) + T_k(\rho_n)\operatorname{div}_{\mathcal{M}_n}\boldsymbol{u}_n + R_n = 0, \qquad \text{in } \mathcal{D}'(\mathbb{R}^3), \quad \forall n \in \mathbb{N}. \tag{5.53}$$

$$\operatorname{div}(L_k(\rho)\boldsymbol{u}) + T_k(\rho)\operatorname{div}\boldsymbol{u} \geq 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^3), \tag{5.54}$$

where the discrete function R_n satisfies: $\displaystyle\int_\Omega R_n \geq 0.$

Proof. To prove (5.53), we apply Proposition 4.8 (completed by Remark 4.3) with the function $b = L_k$ which is a convex function satisfying $|L_k'(t)| \le C|\ln t|$ for $t \le 1$. We straightforwardly obtain (5.53).

Let $M \in \mathbb{N}^*$. Applying Proposition 5.12 to the function $T_M(t)$ (*i.e.* $T_M = b_M$ with $b = Id$) we obtain:

$$\mathrm{div}\big(T_M(\rho)\boldsymbol{u}\big)_K + \big([T_M]'_+(\rho_K)\rho_K - T_M(\rho_K)\big)\,\mathrm{div}(\boldsymbol{u})_K + R_K^1 + R_K^2 + R_K^3 = 0, \quad \forall K \in \mathcal{M}.$$

Let $\phi \in \mathcal{C}_c^\infty(\Omega)$ with $\phi \ge 0$. For $n \in \mathbb{N}$ define $\phi_n \in \mathrm{L}_{\mathcal{M}_n}(\Omega)$ by $\phi_{n|K} = \phi_K$ the mean value of ϕ over K, for $K \in \mathcal{M}_n$. Multiplying the above identify by $|K|\phi_K$ and summing over $K \in \mathcal{M}_n$ yields:

$$\sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)} |\sigma| T_M(\rho_\sigma)(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\phi_K \\ + \sum_{K\in\mathcal{M}_n}\big([T_M]'_+(\rho_K)\rho_K - T_M(\rho_K)\big)\,\phi_K\Big(\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}\Big) + R_1^n + R_2^n + R_3^n = 0,$$

with

$$R_1^n = \sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\Big([T_M]'_+(\rho_K)(\rho_\sigma-\rho_K) + T_M(\rho_K) - T_M(\rho_\sigma)\Big)(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\phi_K,$$

$$R_2^n = h_n^{\xi_2}\sum_{K\in\mathcal{M}_n}[T_M]'_+(\rho_K)\phi_K\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}}|\rho_K-\rho_L|^{\frac{1}{\eta}-1}(\rho_K-\rho_L),$$

$$R_3^n = h_n^{\xi_1}\sum_{K\in\mathcal{M}_n}|K|[T_M]'_+(\rho_K)\phi_K(\rho_K-\rho^\star).$$

Since the function T_M is concave and ρ_σ is the upwind value of the density at the face σ with respect to $\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}$, we have $R_1^n \le 0$. The second remainder term can be rearranged as follows:

$$R_2^n = h_n^{\xi_2}\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}}|\sigma|\left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}}|\rho_K-\rho_L|^{\frac{1}{\eta}-1}(\rho_K-\rho_L)\,([T_M]'_+(\rho_K)\phi_K - [T_M]'_+(\rho_L)\phi_L), \\ = R_{2,1}^n + R_{2,2}^n,$$

where

$$R_{2,1}^n = h_n^{\xi_2}\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}}|\sigma|\left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}}|\rho_K-\rho_L|^{\frac{1}{\eta}-1}(\rho_K-\rho_L)\,([T_M]'_+(\rho_K) - [T_M]'_+(\rho_L))\,\phi_L,$$

$$R_{2,2}^n = h_n^{\xi_2}\sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}}|\sigma|\left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}}|\rho_K-\rho_L|^{\frac{1}{\eta}-1}(\rho_K-\rho_L)\,[T_M]'_+(\rho_K)\,(\phi_K-\phi_L).$$

Since T_M is concave, we have $R_{2,1}^n \le 0$. Hence we get:

$$\sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)} |\sigma| T_M(\rho_\sigma)(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\phi_K \\ + \sum_{K\in\mathcal{M}_n}\big([T_M]'_+(\rho_K)\rho_K - T_M(\rho_K)\big)\,\phi_K\Big(\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}\Big) + R_{2,2}^n + R_3^n \ge 0. \quad (5.55)$$

We want to pass to the limit $n \to +\infty$ in (5.55). To that end, we show that the remainder terms $R^n_{2,2}$ and R^n_3 converge to 0 as $n \to +\infty$. Observing that for all $K \in \mathcal{M}_n$, $|[T_M]'_+(\rho_K)| \le 1$, and since ϕ is a smooth function, we get:

$$\begin{aligned} |R^n_{2,2}| &\le h_n^{\xi_2} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}} |\phi_K-\phi_L| \\ &\lesssim h_n^{\xi_2} \|\boldsymbol{\nabla}\phi\|_{\mathbf{L}^\infty(\Omega)} \sum_{\substack{\sigma\in\mathcal{E}_{n,\mathrm{int}}\\ \sigma=K|L}} |D_\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}} \\ &\lesssim h_n^{\xi_2} \|\boldsymbol{\nabla}\phi\|_{\mathbf{L}^\infty(\Omega)} \||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^1(\Omega)}. \end{aligned}$$

Since $1+\eta>1$, Hölder's inequality yields

$$\||\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)|^{\frac{1}{\eta}}\|_{\mathbf{L}^1(\Omega)} \le C(\Omega,\eta)\|\boldsymbol{\nabla}_{\mathcal{E}_n}(\rho_n)\|^{\frac{1}{\eta}}_{\mathbf{L}^{\frac{1+\eta}{\eta}}(\Omega)} \lesssim h_n^{-\frac{1}{1+\eta}(\xi_2+\frac{5}{4\Gamma}(\frac{3}{1+\eta}+\xi_3))}.$$

Therefore

$$|R^n_{2,2}| \lesssim h_n^{\frac{\eta}{1+\eta}(\xi_2-\frac{5}{4\eta\Gamma}(\frac{3}{1+\eta}+\xi_3))} \longrightarrow 0 \quad \text{as } n\to+\infty.$$

For R^n_3 we may write:

$$|R^n_3| \lesssim h_n^{\xi_1}\ \|\phi\|_{\mathrm{L}^\infty(\Omega)} \sum_{K\in\mathcal{M}_n} |K||\rho_K-\rho^\star| \lesssim 2\,|\Omega|\,\rho^\star\,\|\phi\|_{\mathrm{L}^\infty(\Omega)}\,h_n^{\xi_1}$$

so that $R^n_3 \to 0$ as $n\to+\infty$. Coming back to (5.55), it remains to pass to the limit $n\to+\infty$ in the two terms

$$\sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)} |\sigma| T_M(\rho_\sigma)(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\phi_K$$

$$\text{and}\quad \sum_{K\in\mathcal{M}_n} \big([T_M]'_+(\rho_K)\rho_K - T_M(\rho_K)\big)\,\phi_K\Big(\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}\Big).$$

On the one hand, we have by a discrete integration by parts

$$\sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)} |\sigma| T_M(\rho_\sigma)(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\phi_K = -\int_\Omega (\mathcal{P}_{\mathcal{E}_n}T_M(\rho_n))\,(\Pi_{\mathcal{E}_n}\boldsymbol{u}_n)\cdot\boldsymbol{\nabla}_{\mathcal{E}_n}\phi_n\ \mathrm{d}\boldsymbol{x}.$$

Then, using the same arguments as those to pass to the limit in the discrete weak formulation of the mass equation (see the proof of Proposition 5.2 and replace ρ_n by $T_M(\rho_n)$ which converges to $\overline{T_M(\rho)}$ in $\mathrm{L}^\infty(\Omega)$ weak-* topology), we deduce that

$$\lim_{n\to+\infty}\sum_{K\in\mathcal{M}_n}\sum_{\sigma\in\mathcal{E}(K)} |\sigma| T_M(\rho_\sigma)(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\phi_K = -\int_\Omega \overline{T_M(\rho)}\ \boldsymbol{u}\cdot\boldsymbol{\nabla}\phi\ \mathrm{d}\boldsymbol{x}.$$

This is possible because $(T_M(\rho_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(\Omega)$ (while $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{3(\gamma-1)}$ with $3(\gamma-1)\in(\frac{3}{2},6]$ since $\gamma\in(\frac{3}{2},3]$) and a "weak BV estimate" is available for $T_M(\rho_n)$ thanks to the following inequality (recall that $|T_M(r_1) - T_M(r_2)| \le |r_1 - r_2|$ for all $r_1, r_2 \ge 0$):

$$\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma|\,(T_M(\rho_L) - T_M(\rho_K))^2\,|\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}| \le \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma|\,(\rho_L-\rho_K)^2\,|\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}|.$$

On the other hand, we have:

$$\sum_{K\in\mathcal{M}_n} \big([T_M]'_+(\rho_K)\rho_K - T_M(\rho_K)\big)\,\phi_K\Big(\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}\Big) = \int_\Omega \big([T_M]'_+(\rho_n)\rho_n - T_M(\rho_n)\big)\,{\rm div}_{\mathcal{M}_n}\boldsymbol{u}_n\,\phi\,{\rm d}\boldsymbol{x}.$$

Hence, passing to the limit $n\to+\infty$ in (5.55) we obtain:

$$\operatorname{div}\big(\overline{T_M(\rho)}\boldsymbol{u}\big) + \overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]{\rm div}_{\mathcal{M}}\,\boldsymbol{u}} \ge 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3) \tag{5.56}$$

which corresponds to a relaxed version of Equation (2.36) from Section 2. For $k\in\mathbb{N}^*$ and $\delta>0$, we introduce the regularized function $L_{k,\delta}$ defined as $L_{k,\delta}(t) = L_k(t+\delta)$, the derivative of which is bounded close to 0 unlike L_k. Applying Lemma 2.1 (and the second part of Remark 2.4) to the pair $(\overline{T_M(\rho)},\boldsymbol{u})$ (justified since $\overline{T_M(\rho)}\in L^\infty(\Omega)$ for M fixed) with the function $L_{k,\delta}$ and the source term $g = -\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]{\rm div}_{\mathcal{M}}\,\boldsymbol{u}} \in {\rm L}^1_{\rm loc}(\mathbb{R}^3)$, we get:

$$\operatorname{div}\big(L_{k,\delta}\big(\overline{T_M(\rho)}\big)\boldsymbol{u}\big) + T_{k,\delta}\big(\overline{T_M(\rho)}\big)\operatorname{div}\boldsymbol{u} \ge -L'_{k,\delta}\big(\overline{T_M(\rho)}\big)\,\overline{\big[\rho[T_M]'_+(\rho) - T_M(\rho)\big]{\rm div}_{\mathcal{M}}\,\boldsymbol{u}} \quad \text{in } \mathcal{D}'(\mathbb{R}^3) \tag{5.57}$$

where $T_{k,\delta}(t) = tL'_{k,\delta}(t) - L_{k,\delta}(t)$. Now, exactly as in the continuous case, we pass to the limits $M\to+\infty$ and then $\delta\to0^+$ (see the proof of Prop. 2.12) to get inequality (5.54). □

Strong convergence of the density

Proposition 5.14. *Under the assumptions of Theorem 3.1, let* $(\rho,\boldsymbol{u})\in{\rm L}^{3(\gamma-1)}(\Omega)\times\mathbf{H}^1_0(\Omega)$ *be the limit couple of the sequence* $(\rho_n,\boldsymbol{u}_n)_{n\in\mathbb{N}}$*. Up to extraction, the sequence* $(\rho_n)_{n\in\mathbb{N}}$ *strongly converges towards* ρ *in* ${\rm L}^q(\Omega)$ *for all* $q\in[1,3(\gamma-1))$*.*

Proof. Integrating inequalities (5.53) and (5.54) and summing, one obtains:

$$\int_\Omega T_k(\rho_n){\rm div}_{\mathcal{M}_n}\boldsymbol{u}_n\,{\rm d}\boldsymbol{x} - \int_\Omega T_k(\rho)\operatorname{div}\boldsymbol{u}\,{\rm d}\boldsymbol{x} \le 0, \qquad \forall n\in\mathbb{N}. \tag{5.58}$$

Since, $|T_k(r_1)-T_k(r_2)|^{\gamma+1} \le (r_1^\gamma - r_2^\gamma)(T_k(r_1)-T_k(r_2))$, for all r_1, $r_2\ge0$, we have

$$\limsup_{n\to+\infty}\int_\Omega |T_k(\rho_n)-T_k(\rho)|^{\gamma+1}\,{\rm d}\boldsymbol{x} \le \limsup_{n\to+\infty}\int_\Omega (\rho_n^\gamma-\rho^\gamma)(T_k(\rho_n)-T_k(\rho))\,{\rm d}\boldsymbol{x} \le \int_\Omega \big(\overline{\rho^\gamma T_k(\rho)} - \overline{\rho^\gamma}\;\overline{T_k(\rho)}\big)\,{\rm d}\boldsymbol{x} + \int_\Omega \big(\overline{\rho^\gamma}-\rho^\gamma\big)\big(\overline{T_k(\rho)}-T_k(\rho)\big)\,{\rm d}\boldsymbol{x}.$$

Invoking the convexity of the functions $t \mapsto t^\gamma$ and $t \mapsto -T_k(t)$, we have $\overline{\rho^\gamma} \geq \rho^\gamma$ and $\overline{T_k(\rho)} \leq T_k(\rho)$ so that

$$\limsup_{n\to+\infty} \int_\Omega |T_k(\rho_n) - T_k(\rho)|^{\gamma+1} \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \big(\overline{\rho^\gamma T_k(\rho)} - \overline{\rho^\gamma}\ \overline{T_k(\rho)}\big) \,\mathrm{d}\boldsymbol{x}.$$

We can now use the weak compactness property satisfied by the effective viscous flux (Prop. 5.7):

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_\Omega |T_k(\rho_n) - T_k(\rho)|^{\gamma+1} \,\mathrm{d}\boldsymbol{x} \\
&\leq \frac{2\mu+\lambda}{a} \limsup_{n\to+\infty} \int_\Omega \big(T_k(\rho_n) - \overline{T_k(\rho)}\big) \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} \\
&= \frac{2\mu+\lambda}{a} \int_\Omega (T_k(\rho) - \overline{T_k(\rho)})\, \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} + \limsup_{n\to+\infty} \Big(\int_\Omega T_k(\rho_n) \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} - \int_\Omega T_k(\rho) \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}\Big) \\
&\leq \frac{2\mu+\lambda}{a} \int_\Omega (T_k(\rho) - \overline{T_k(\rho)})\, \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

thanks to (5.58). The end of the proof is the same as that of Proposition 2.13: thanks to the previous inequality we show that

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \int_\Omega |T_k(\rho_n) - T_k(\rho)|^{\gamma+1} \,\mathrm{d}\boldsymbol{x} = 0,$$

and thus

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \|T_k(\rho_n) - T_k(\rho)\|_{\mathrm{L}^1(\Omega)} = 0.$$

We conclude to the strong convergence of the density by passing to the limits $n \to +\infty$, $k \to +\infty$ in the following inequality

$$\|\rho - \rho_n\|_{\mathrm{L}^1(\Omega)} \leq \|\rho_n - T_k(\rho_n)\|_{\mathrm{L}^1(\Omega)} + \|T_k(\rho_n) - T_k(\rho)\|_{\mathrm{L}^1(\Omega)} + \|T_k(\rho) - \rho\|_{\mathrm{L}^1(\Omega)}.$$

□

A Proof of some technical lemmas

A.1 The discrete renormalization property

Proposition A.1 (Discrete renormalization property)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. Let $(\rho, \boldsymbol{u}) \in \mathrm{L}_\mathcal{M}(\Omega) \times \mathbf{H}_{\mathcal{M},0}(\Omega)$ satisfy the discrete mass balance* (3.4a)*. We have $\rho > 0$* a.e. *in Ω (*i.e. $\rho_K > 0$, $\forall K \in \mathcal{M}$*). Then, for any $b \in \mathcal{C}^1([0,+\infty))$:*

$$\mathrm{div}\big(b(\rho)\boldsymbol{u}\big)_K + \big(b'(\rho_K)\rho_K - b(\rho_K)\big)\mathrm{div}(\boldsymbol{u})_K + R_K^1 + R_K^2 + R_K^3 = 0 \quad \forall K \in \mathcal{M}, \tag{A.1}$$

where

$$\mathrm{div}\big(b(\rho)\boldsymbol{u}\big)_K = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\ b(\rho_\sigma)\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma},$$

and

$$R_K^1 = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r_{K,\sigma} \, (\boldsymbol{u}_\sigma \cdot n_{K,\sigma}) \quad \textit{and} \quad r_{K,\sigma} = b'(\rho_K)(\rho_\sigma - \rho_K) + b(\rho_K) - b(\rho_\sigma),$$

$$R_K^2 = h_{\mathcal{M}}^{\xi_2} \, b'(\rho_K) \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L),$$

$$R_K^3 = h_{\mathcal{M}}^{\xi_1} b'(\rho_K)(\rho_K - \rho^\star).$$

Multiplying by $|K|$ *and summing over* $K \in \mathcal{M}$*, it holds*

$$\int_\Omega \big(b'(\rho)\rho - b(\rho)\big) \mathrm{div}_{\mathcal{M}} \, \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} + R_{\mathcal{E}}^1 + R_{\mathcal{E}}^2 + R_{\mathcal{M}}^3 = 0, \tag{A.2}$$

with

$$R_{\mathcal{E}}^1 = \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}} \\ \sigma=K|L}} |\sigma| (r_{K,\sigma} - r_{L,\sigma}) \, (\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}),$$

$$R_{\mathcal{E}}^2 = h_{\mathcal{M}}^{\xi_2} \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}} \\ \sigma=K|L}} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L)(b'(\rho_K) - b'(\rho_L)),$$

$$R_{\mathcal{M}}^3 = h_{\mathcal{M}}^{\xi_1} \sum_{K\in\mathcal{M}} |K| b'(\rho_K)(\rho_K - \rho^\star).$$

Moreover, if b *is convex then* $R_{\mathcal{E}}^{1,2} \geq 0$ *and* $R_{\mathcal{M}}^3 \geq 0$*.*

Proof. Multiplying by $b'(\rho_K)\mathcal{X}_K$ the discrete mass conservation equation (3.4a) (together with the definition (3.6)), one gets

$$b'(\rho_K) \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \rho_\sigma \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} + h_{\mathcal{M}}^{\xi_1} b'(\rho_K)(\rho_K - \rho^\star)$$
$$+ h_{\mathcal{M}}^{\xi_2} b'(\rho_K) \frac{1}{|K|} \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}} \\ \sigma=K|L}} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L) = 0.$$

and then

$$\frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| b(\rho_\sigma) \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} + \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \big(b'(\rho_K)\rho_K - b(\rho_K)\big) \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}$$
$$+ \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| r_{K,\sigma} \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}$$
$$+ h_{\mathcal{M}}^{\xi_2} b'(\rho_K) \frac{1}{|K|} \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}} \\ \sigma=K|L}} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L)$$
$$+ h_{\mathcal{M}}^{\xi_1} b'(\rho_K)(\rho_K - \rho^\star) = 0$$

with

$$r_{K,\sigma} = b'(\rho_K)(\rho_\sigma - \rho_K) + b(\rho_K) - b(\rho_\sigma),$$

which corresponds to Equation (A.1). Multiplying by $|K|$, summing over K rearranging the sums, and using the discrete homogeneous Dirichlet boundary condition on the velocity, we get (A.2).

Let us assume from now on that b is convex, First, we have $r_{K,\sigma} = 0$ when $\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} \geq 0$ (since then $\rho_\sigma = \rho_K$) and, when $\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} \leq 0$, we have $r_{K,\sigma} \leq 0$ since b is convex. Hence $R^1_{\mathcal{E}} \geq 0$. Since b is convex, we also deduce that

$$R^2_{\mathcal{E}} = h_{\mathcal{M}}^{\xi_2} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}\Big)^{\frac{1}{\eta}} |\rho_K - \rho_L|^{\frac{1}{\eta}-1} (\rho_K - \rho_L)(b'(\rho_K) - b'(\rho_L)) \geq 0.$$

Finally for the last remainder term $R^3_{\mathcal{M}}$, we combine the convexity of b with a Taylor expansion and then use Jensen's inequality (recalling that $\sum_{K\in\mathcal{M}} |K|\,\rho_K = |\Omega|\rho^\star$) to get:

$$R^3_{\mathcal{M}} \geq h_{\mathcal{M}}^{\xi_1} \sum_{K\in\mathcal{M}} |K|\,(b(\rho_K) - b(\rho^\star)) = h_{\mathcal{M}}^{\xi_1}\,|\Omega| \left(\frac{1}{|\Omega|}\int_\Omega b(\rho)\,\mathrm{d}\boldsymbol{x} - b\Big(\frac{1}{|\Omega|}\int_\Omega \rho\,\mathrm{d}\boldsymbol{x}\Big)\right) \geq 0.$$

□

A.2 Estimate on the momentum convection term

Lemma A.2. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω in the sense of Definition 3.1, such that $\theta_{\mathcal{M}} \leq \theta_0$. Define*

$$R_{\rm conv}(\rho, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathbf{div}_{\mathcal{E}}(\rho\boldsymbol{u}\otimes\boldsymbol{u})\cdot\Pi_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega (\mathcal{P}_{\mathcal{E}}\rho)(\Pi_{\mathcal{E}}\boldsymbol{u})\otimes(\Pi_{\mathcal{E}}\boldsymbol{u}) : \boldsymbol{\nabla}_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}.$$

with the operators $\boldsymbol{\nabla}_{\mathcal{E}}$, $\mathcal{P}_{\mathcal{E}}$ and $\Pi_{\mathcal{E}}$ defined in (4.5), (4.8) and (4.4). Assume $h_{\mathcal{M}} \leq 1$. Then, there exists $C = C(\Omega, \gamma, \Gamma, \theta_0)$ such that for all $(\rho, \boldsymbol{u}, \boldsymbol{v}) \in \mathrm{L}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M}}(\Omega) \times \mathbf{H}_{\mathcal{M}}(\Omega)$:

$$\big|R_{\rm conv}(\rho, \boldsymbol{u}, \boldsymbol{v})\big| \leq C\,h_{\mathcal{M}}^{\frac{1}{2}-\frac{1}{\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \|h^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\Gamma}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}}^2\,\|\boldsymbol{v}\|_{1,2,\mathcal{M}} \tag{A.3}$$

$$+ C\,h_{\mathcal{M}}^{\xi_2-\frac{1}{\eta}-\frac{1}{\eta\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \|h^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\eta\Gamma}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}}\,\|\boldsymbol{v}\|_{1,2,\mathcal{M}}.$$

Proof. By definition, recalling that $\boldsymbol{a}\otimes\boldsymbol{b} : \boldsymbol{c}\otimes\boldsymbol{d} = (\boldsymbol{a}\cdot\boldsymbol{c})\,(\boldsymbol{b}\cdot\boldsymbol{d})$ for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$, we have:

$$\int_\Omega (\mathcal{P}_{\mathcal{E}}\rho)(\Pi_{\mathcal{E}}\boldsymbol{u})\otimes(\Pi_{\mathcal{E}}\boldsymbol{u}) : \boldsymbol{\nabla}_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |D_\sigma|\,\rho_\sigma\,(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\,\boldsymbol{u}_\sigma\cdot\Big(\frac{|\sigma|}{|D_\sigma|}(\boldsymbol{v}_L - \boldsymbol{v}_K)\Big).$$

Reordering the sum and using the definition of the primal fluxes (3.6) we get:

$$-\int_\Omega (\mathcal{P}_{\mathcal{E}}\rho)(\Pi_{\mathcal{E}}\boldsymbol{u})\otimes(\Pi_{\mathcal{E}}\boldsymbol{u}) : \boldsymbol{\nabla}_{\mathcal{E}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathcal{M}} \boldsymbol{v}_K \cdot \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\rho_\sigma\,(\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma})\,\boldsymbol{u}_\sigma$$

$$= \sum_{K\in\mathcal{M}} \boldsymbol{v}_K \cdot \sum_{\sigma\in\mathcal{E}(K)} \overline{F}_{K,\sigma}(\rho, \boldsymbol{u})\,\boldsymbol{u}_\sigma + R_1$$

where

$$R_1 = -h_{\mathcal{M}}^{\xi_2} \sum_{K\in\mathcal{M}} \boldsymbol{v}_K \cdot \sum_{\substack{\sigma\in\mathcal{E}(K)\\ \sigma=K|L}} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}-1} (\rho_K-\rho_L)\, \boldsymbol{u}_\sigma.$$

By assumption $(H2)$ (conservativity of the dual fluxes) we may write :

$$\begin{aligned}
-\int_\Omega (\mathcal{P}_\mathcal{E}\rho)(\Pi_\mathcal{E}\boldsymbol{u})\otimes(\Pi_\mathcal{E}\boldsymbol{u}) : \boldsymbol{\nabla}_\mathcal{E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}&\\
&= \sum_{K\in\mathcal{M}} \boldsymbol{v}_K \cdot \sum_{\sigma\in\mathcal{E}(K)} \overline{F}_{K,\sigma}(\rho,\boldsymbol{u})\,\boldsymbol{u}_\sigma + R_1\\
&= \sum_{K\in\mathcal{M}} \boldsymbol{v}_K \cdot \sum_{\sigma\in\mathcal{E}(K)} \Big(\overline{F}_{K,\sigma}(\rho,\boldsymbol{u})\ \boldsymbol{u}_\sigma + \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\,\epsilon\subset K} F_{\sigma,\epsilon}(\rho,\boldsymbol{u})\,\boldsymbol{u}_\epsilon\Big) + R_1.
\end{aligned}$$

Writing $\boldsymbol{v}_K = \boldsymbol{v}_\sigma + \boldsymbol{v}_K - \boldsymbol{v}_\sigma$ we get:

$$\begin{aligned}
-\int_\Omega (\mathcal{P}_\mathcal{E}\rho)(\Pi_\mathcal{E}\boldsymbol{u})\otimes(\Pi_\mathcal{E}\boldsymbol{u}) : \boldsymbol{\nabla}_\mathcal{E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}&\\
&= \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} \boldsymbol{v}_\sigma \cdot \Big(\overline{F}_{K,\sigma}(\rho,\boldsymbol{u})\ \boldsymbol{u}_\sigma + \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\,\epsilon\subset K} F_{\sigma,\epsilon}(\rho,\boldsymbol{u})\,\boldsymbol{u}_\epsilon\Big) + R_1 + R_2,
\end{aligned} \tag{A.4}$$

with

$$R_2 = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} (\boldsymbol{v}_K-\boldsymbol{v}_\sigma) \cdot \Big(\overline{F}_{K,\sigma}(\rho,\boldsymbol{u})\ \boldsymbol{u}_\sigma + \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\,\epsilon\subset K} F_{\sigma,\epsilon}(\rho,\boldsymbol{u})\,\boldsymbol{u}_\epsilon\Big).$$

By conservativity of the primal fluxes (*i.e.* using $\overline{F}_{K,\sigma}(\rho,\boldsymbol{u}) = -\overline{F}_{L,\sigma}(\rho,\boldsymbol{u})$ for $\sigma = K|L$) we see that the first term in the right hand side of (A.4) is equal to $\int_\Omega \mathbf{div}_\mathcal{E}(\rho\boldsymbol{u}\otimes\boldsymbol{u})\cdot\Pi_\mathcal{E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$. Hence:

$$\Big|\int_\Omega \mathbf{div}_\mathcal{E}(\rho\boldsymbol{u}\otimes\boldsymbol{u})\cdot\Pi_\mathcal{E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega (\mathcal{P}_\mathcal{E}\rho)(\Pi_\mathcal{E}\boldsymbol{u})\otimes(\Pi_\mathcal{E}\boldsymbol{u}) : \boldsymbol{\nabla}_\mathcal{E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\Big| \le |R_1| + |R_2|.$$

Proving Lemma A.2 amounts to bounding $|R_1|$ and $|R_2|$. We begin with $|R_1|$. Reordering the sum in R_1 we, get for $C = (\Omega,\gamma,\Gamma,\theta_0)$:

$$\begin{aligned}
|R_1| &= h_{\mathcal{M}}^{\xi_2} \Big| \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} |\sigma| \left(\frac{|\sigma|}{|D_\sigma|}\right)^{\frac{1}{\eta}} |\rho_K-\rho_L|^{\frac{1}{\eta}-1} (\rho_K-\rho_L)\,\boldsymbol{u}_\sigma\cdot(\boldsymbol{v}_K-\boldsymbol{v}_L)\Big|\\
&\le C\, h_{\mathcal{M}}^{\xi_2-\frac{1}{\eta}} \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}} \Big(\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |D_\sigma||\boldsymbol{u}_\sigma|^6\Big)^{\frac{1}{6}} \Bigg(\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |D_\sigma| \Big(\frac{|\sigma|}{|D_\sigma|}|\boldsymbol{v}_K-\boldsymbol{v}_L|\Big)^{\frac{6}{5}}\Bigg)^{\frac{5}{6}}\\
&\le C\, h_{\mathcal{M}}^{\xi_2-\frac{1}{\eta}} \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}} \|\Pi_\mathcal{E}\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} \|\boldsymbol{\nabla}_\mathcal{E}\boldsymbol{v}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)^3}\\
&\le C\, h_{\mathcal{M}}^{\xi_2-\frac{1}{\eta}} \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}} \|\boldsymbol{\nabla}_\mathcal{E}\boldsymbol{v}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)^3}\\
&\le C\, h_{\mathcal{M}}^{\xi_2-\frac{1}{\eta}-\frac{3}{(1+\eta)\eta\Gamma}} \|\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\eta\Gamma}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}} \|\boldsymbol{\nabla}_\mathcal{E}\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)^3}\\
&\le C\, h_{\mathcal{M}}^{\xi_2-\frac{1}{\eta}-\frac{1}{\eta\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \|h_{\mathcal{M}}^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\eta\Gamma}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}} \|\boldsymbol{v}\|_{1,2,\mathcal{M}}.
\end{aligned}$$

Let us now turn to R_2. Recalling that $\boldsymbol{u}_\epsilon = \boldsymbol{u}_\sigma + \frac{1}{2}(\boldsymbol{u}_{\sigma'} - \boldsymbol{u}_\sigma)$ and using $(H1)$, we write $R_2 = R_{2,1} + R_{2,2}$ with:

$$R_{2,1} = \frac{1}{2} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} (\boldsymbol{v}_K - \boldsymbol{v}_\sigma) \cdot \Big(\sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\\ \epsilon\subset K,\ \epsilon=D_\sigma|D'_\sigma}} F_{\sigma,\epsilon}(\rho,\boldsymbol{u})\,(\boldsymbol{u}_{\sigma'} - \boldsymbol{u}_\sigma)\Big),$$

$$R_{2,2} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} (\boldsymbol{v}_K - \boldsymbol{v}_\sigma) \cdot\ \boldsymbol{u}_\sigma\ \xi_K^\sigma \Big(\sum_{\sigma'\in\mathcal{E}(K)} \overline{F}_{K,\sigma'}(\rho,\boldsymbol{u})\Big).$$

The assumption $(H3)$ yields, for $C = C(\Omega,\theta_0)$:

$$\begin{aligned}|F_{\sigma,\epsilon}(\rho,\boldsymbol{u})| &\leq C\Big(\|\rho\|_{\mathrm{L}^\infty(\Omega)}\|\Pi_{\mathcal{E}}\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)}\ h_K^2 + \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}}\, h_{\mathcal{M}}^{\xi_2+1-\frac{1}{\eta}}\, h_K\Big)\\ &\leq C\Big(\|\rho\|_{\mathrm{L}^\infty(\Omega)}\|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)}\, h_{\mathcal{M}}\, h_K + \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}}\, h_{\mathcal{M}}^{\xi_2+1-\frac{1}{\eta}}\, h_K\Big).\end{aligned}$$

Since $\boldsymbol{v}_K$ is a convex combination of $(\boldsymbol{v}_\sigma)_{\sigma\in\mathcal{E}(K)}$:

$$\begin{aligned}&\Big|\sum_{\sigma\in\mathcal{E}(K)} (\boldsymbol{v}_K - \boldsymbol{v}_\sigma)\cdot\Big(\sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma),\\ \epsilon\subset K,\ \epsilon=D_\sigma|D'_\sigma}} F_{\sigma,\epsilon}(\rho,\boldsymbol{u})\,(\boldsymbol{u}_{\sigma'} - \boldsymbol{u}_\sigma)\Big)\Big|\\ &\quad\leq C\Big(\|\rho\|_{\mathrm{L}^\infty(\Omega)}\|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)}\ h_{\mathcal{M}} + \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}}\, h_{\mathcal{M}}^{\xi_2+1-\frac{1}{\eta}}\Big) \sum_{\substack{\sigma,\sigma'\\ \sigma'',\sigma'''\in\mathcal{E}(K)}} h_K\ |\boldsymbol{v}_\sigma - \boldsymbol{v}_{\sigma'}|\ |\boldsymbol{u}_{\sigma''} - \boldsymbol{u}_{\sigma'''}|,\end{aligned}$$

and, for $\sigma,\ \sigma'\in\mathcal{E}(K)$, the quantity $|\boldsymbol{u}_\sigma - \boldsymbol{u}_{\sigma'}|$ (or $|\boldsymbol{v}_\sigma - \boldsymbol{v}_{\sigma'}|$) appears in the sum a finite number of times which depends on the number of faces of K. Hence, applying the Cauchy-Schwarz inequality and Lemma 4.4, we may write

$$\begin{aligned}|R_{2,1}| &\leq C\Big(\|\rho\|_{\mathrm{L}^\infty(\Omega)}\|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)}\ h_{\mathcal{M}} + \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}}\, h_{\mathcal{M}}^{\xi_2+1-\frac{1}{\eta}}\Big)\,\|\boldsymbol{u}\|_{1,2,\mathcal{E}}\,\|\boldsymbol{v}\|_{1,2,\mathcal{E}}\\ &\leq C\Big(\|\rho\|_{\mathrm{L}^\infty(\Omega)}\|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)}\ h_{\mathcal{M}} + \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}}\, h_{\mathcal{M}}^{\xi_2+1-\frac{1}{\eta}}\Big)\,\|\boldsymbol{u}\|_{1,2,\mathcal{M}}\,\|\boldsymbol{v}\|_{1,2,\mathcal{M}}\end{aligned} \tag{A.5}$$

We then get, for $C = C(\Omega,\gamma,\Gamma,\theta_0)$:

$$\begin{aligned}|R_{2,1}| \leq\ & C\, h_{\mathcal{M}}^{1-\frac{1}{2}-\frac{1}{\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)}\,\|h_{\mathcal{M}}^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\Gamma}}\|\boldsymbol{u}\|_{1,2,\mathcal{M}}^2\,\|\boldsymbol{v}\|_{1,2,\mathcal{M}}\\ &+ C\, h_{\mathcal{M}}^{\xi_2+1-\frac{1}{\eta}-\frac{1}{\eta\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)}\,\|h_{\mathcal{M}}^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\eta\Gamma}}\|\boldsymbol{u}\|_{1,2,\mathcal{M}}\,\|\boldsymbol{v}\|_{1,2,\mathcal{M}}.\end{aligned}$$

The estimation of $R_{2,2}$ follows similar steps. Indeed by definition of $\boldsymbol{v}_K$, we have

$$\sum_{\sigma\in\mathcal{E}(K)} \xi_K^\sigma\ (\boldsymbol{v}_K - \boldsymbol{v}_\sigma) = 0,$$

and we obtain that:

$$R_{2,2} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} (\boldsymbol{v}_K - \boldsymbol{v}_\sigma)\cdot \xi_K^\sigma \ (\boldsymbol{u}_\sigma - \boldsymbol{u}_K) \Big[\sum_{\sigma'\in\mathcal{E}(K)} \overline{F}_{K,\sigma'}(\rho,\boldsymbol{u})\Big],$$

so, once again, denoting $\boldsymbol{u}_K = \sum_{\sigma\in\mathcal{E}(K)} \xi_K^\sigma \ \boldsymbol{u}_\sigma$:

$$|R_{2,2}| \leq C\Big(\|\rho\|_{\mathrm{L}^\infty(\Omega)} \|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)} \ h_\mathcal{M} + \|\rho\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{\eta}} \, h_\mathcal{M}^{\xi_2+1-\frac{1}{\eta}}\Big) \sum_{K\in\mathcal{M}} h_K \sum_{\sigma\in\mathcal{E}(K)} |\boldsymbol{v}_\sigma - \boldsymbol{v}_K| \ |\boldsymbol{u}_\sigma - \boldsymbol{u}_K|$$

$$\leq C\, h_\mathcal{M}^{1-\frac{1}{2}-\frac{1}{\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \|h_\mathcal{M}^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\Gamma}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}}^2 \, \|\boldsymbol{v}\|_{1,2,\mathcal{M}}$$

$$+ \, C\, h_\mathcal{M}^{\xi_2+1-\frac{1}{\eta}-\frac{1}{\eta\Gamma}\left(\frac{3}{1+\eta}+\xi_3\right)} \|h_\mathcal{M}^{\xi_3}\rho^\Gamma\|_{\mathrm{L}^{1+\eta}(\Omega)}^{\frac{1}{\eta\Gamma}} \|\boldsymbol{u}\|_{1,2,\mathcal{M}} \, \|\boldsymbol{v}\|_{1,2,\mathcal{M}}.$$

□

B A topological degree result

The following theorem follows from standard arguments of the topological degree theory (see [6] for an overview of theory and *e.g.* [10, 19, 32] for other uses in the same objective as here, namely the proof of existence of a solution to a numerical scheme).

Theorem B.1. *Let N and M be two positive integers and $V = \mathbb{R}^N \times \mathbb{R}^M$. Let $b \in V$ and $f(\cdot)$ and $\mathcal{F}(\cdot,\cdot)$ be two continuous functions respectively from V and $V\times[0,1]$ to V. Assume that:*

(i) $\mathcal{F}(\cdot,1) = f(\cdot)$*;*

(ii) $\forall\delta\in[0,1]$*, if an element v of $\bar{\mathcal{O}}$ (the closure of $\mathcal{O}$) is such that $\mathcal{F}(v,\delta) = b$, then $v\in\mathcal{O}$, where $\mathcal{O}$ is defined as follows:*

$$\mathcal{O} = \{(x,y)\in V \ \textit{s.t.} \ C_0 < x < C_1 \ \textit{and} \ \|y\|_M < C_2\}$$

where, for any real number c and vector x, the notation $x > c$ means that each component of x is larger than c; C_0, C_1 and C_2 being positive constants and $\|y\|_M$ a norm defined on $\mathbb{R}^M$;

(iii) the topological degree of $\mathcal{F}(\cdot,0)$ with respect to b and $\mathcal{O}$ is equal to $d_0 \neq 0$.

Then the topological degree of $\mathcal{F}(\cdot,1)$ with respect to b and $\mathcal{O}$ is also equal to $d_0\neq 0$; consequently, there exists at least one solution $v\in\mathcal{O}$ to the equation $f(v) = b$.

C Discrete functional analysis for non-conforming finite elements

In this appendix, we prove that some important functional analysis results can be extended to piecewise smooth functions obtained by non-conforming finite element approximations. We focus on the Crouzeix-Raviart finite elements [5] but all the results can be extended to the Rannacher-Turek

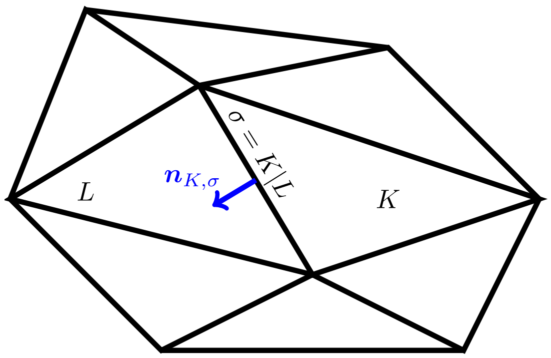

Figure 2: Notations for triangular meshes.

finite elements [37] which use the same degrees of freedom (except the approximation property (C.16) which must be adapted, see [37]). We prove classical results such as a discrete *inf-sup* property as well as well known approximation results. We also prove discrete Sobolev inequalities as well as compactness results which are discrete counterparts to Rellich's theorem. The proof of these last properties are widely based on the work by R. Eymard, T. Gallouët, R. Herbin, and their collaborators. We refer to the books [11, 8] and also to the appendix in [12] where similar results are proven for finite volume schemes.

C.1 Meshes and discrete functions

Let Ω be an open bounded connected subset of $\mathbb{R}^d$, $d \in \{2,3\}$. We assume that Ω is polygonal if $d = 2$ and polyhedral if $d = 3$. We define triangular meshes in the following way.

Definition C.1 (Triangular mesh (see Figure 2))**.** *A triangulation (or triangular mesh) of Ω is a finite family $\mathcal{M}$ composed of non empty simplices such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For any simplex $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K, which is the union of cell faces. We denote by $\mathcal{E}$ the set of faces of the mesh, and we suppose that two neighboring cells share a whole face: for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K, L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \overline{\sigma}$; we denote in the latter case $\sigma = K|L$. We denote by $\mathcal{E}_{\text{ext}}$ and $\mathcal{E}_{\text{int}}$ the set of external and internal faces: $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$ and $\mathcal{E}_{\text{int}} = \mathcal{E} \setminus \mathcal{E}_{\text{ext}}$. For $K \in \mathcal{M}$, $\mathcal{E}(K)$ stands for the set of faces of K. The unit vector normal to $\sigma \in \mathcal{E}(K)$ outward K is denoted by $\boldsymbol{n}_{K,\sigma}$. In the following, the notation $|K|$ or $|\sigma|$ stands indifferently for the d-dimensional or the $(d-1)$-dimensional measure of the subset K of $\mathbb{R}^d$ or σ of $\mathbb{R}^{d-1}$ respectively.*

Definition C.2 (Size of the discretization)**.** *Let $\mathcal{M}$ be a triangulation of Ω. For every $K \in \mathcal{M}$, we denote h_K the diameter of K (*i.e.* the 1D measure of the largest line segment included in K) and for every $\sigma \in \mathcal{E}$, we denote h_σ the diameter of σ. The size of the discretization is defined by:*

$$h_{\mathcal{M}} = \max_{K\in\mathcal{M}} h_K.$$

Definition C.3 (Regularity of the discretization)**.** *Let $\mathcal{M}$ be a triangulation of Ω. For every $K \in \mathcal{M}$, denote ϱ_K the radius of the largest ball included in K. The regularity parameter of the discretization is defined by:*

$$\theta_{\mathcal{M}} = \max\Big\{\frac{h_K}{\varrho_K},\, K \in \mathcal{M}\Big\}. \tag{C.1}$$

Definition C.4. *Let $\mathcal{M}$ be a triangulation of Ω. We denote $\mathrm{L}_{\mathcal{M}}(\Omega)$ the space of scalar functions that are constant on each primal cell $K \in \mathcal{M}$. For $p \in \mathrm{L}_{\mathcal{M}}(\Omega)$ and $K \in \mathcal{M}$, we denote p_K the constant value of p on K. We denote $\mathrm{L}_{\mathcal{M},0}(\Omega)$ the subspace of $\mathrm{L}_{\mathcal{M}}(\Omega)$ composed of zero average functions over Ω.*

Let $P_1(K)$ be the space of degree one polynomials defined over K:

$$P_1(K) = \mathrm{span}\Big\{1,\, x_i,\, i = 1, .., d\Big\}.$$

Definition C.5. *Let $\mathcal{M}$ be a triangulation of Ω. We denote $\mathrm{H}_{\mathcal{M}}(\Omega)$ the space of functions u such that $u_{|K} \in P_1(K)$ for all $K \in \mathcal{M}$ and such that:*

$$\frac{1}{|\sigma|}\int_\sigma [u]_\sigma \,\mathrm{d}\sigma(\boldsymbol{x}) = 0, \qquad \forall \sigma \in \mathcal{E}_{\mathrm{int}}, \tag{C.2}$$

where $[u]_\sigma$ is the jump of u through σ which is defined on $\sigma = K|L$ by $[u]_\sigma = u_{|L} - u_{|K}$. We define $\mathrm{H}_{\mathcal{M},0}(\Omega) \subset \mathrm{H}_{\mathcal{M}}(\Omega)$ the subspace of $\mathrm{H}_{\mathcal{M}}(\Omega)$ composed of functions $u \in \mathrm{H}_{\mathcal{M}}(\Omega)$ such that $\frac{1}{|\sigma|}\int_\sigma u \,\mathrm{d}\sigma(\boldsymbol{x}) = 0$ for all $\sigma \in \mathcal{E}_{\mathrm{ext}}$. Finally, we denote $\mathbf{H}_{\mathcal{M}}(\Omega) := \mathrm{H}_{\mathcal{M}}(\Omega)^d$ and $\mathbf{H}_{\mathcal{M},0}(\Omega) := \mathrm{H}_{\mathcal{M},0}(\Omega)^d$.

For a discrete field $u \in \mathrm{H}_{\mathcal{M}}(\Omega)$ and $\sigma \in \mathcal{E}$, the degree of freedom associated with σ is given by:

$$u_\sigma = \frac{1}{|\sigma|}\int_\sigma u \,\mathrm{d}\sigma(\boldsymbol{x}). \tag{C.3}$$

Although $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$ is discontinuous across an internal face $\sigma \in \mathcal{E}_{\mathrm{int}}$, the definition of u_σ is unambiguous thanks to (C.2).

Definition C.6 (Shape functions)**.** *Let $\mathcal{M}$ be a triangulation of Ω. The shape function $\zeta_\sigma \in \mathrm{H}_{\mathcal{M}}(\Omega)$ associated with $\sigma \in \mathcal{E}$, is the unique function in $\mathrm{H}_{\mathcal{M}}(\Omega)$ satisfying for all $\sigma, \sigma' \in \mathcal{E}$:*

$$\frac{1}{|\sigma'|}\int_{\sigma'} \zeta_\sigma \,\mathrm{d}\sigma(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \sigma' = \sigma, \\ 0, & \text{if } \sigma' \neq \sigma. \end{cases}$$

Then for all $u \in \mathrm{H}_{\mathcal{M}}(\Omega)$:

$$u = \sum_{\sigma \in \mathcal{E}} u_\sigma\, \zeta_\sigma, \qquad \text{where} \quad u_\sigma = \frac{1}{|\sigma|}\int_\sigma u \,\mathrm{d}\sigma(\boldsymbol{x}), \quad \forall \sigma \in \mathcal{E}.$$

Observe that the support of a shape function $\zeta_\sigma \in \mathrm{H}_{\mathcal{M}}(\Omega)$ is included in the (at most) two neighboring cells to σ.

Definition C.7 (Discrete $W^{1,q}$ semi-norm)**.** *Let $\mathcal{M}$ be a triangulation of Ω. We define the piecewise smooth gradient and divergence operators acting on discrete functions $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$:*

$$\boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{u}(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} \boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{x}) \mathcal{X}_K(\boldsymbol{x}), \tag{C.4}$$

$$\operatorname{div}_{\mathcal{M}} \boldsymbol{u}(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} \operatorname{div} \boldsymbol{u}(\boldsymbol{x}) \mathcal{X}_K(\boldsymbol{x}). \tag{C.5}$$

For all scalar functions $u \in \mathrm{H}_{\mathcal{M}}(\Omega)$ and all $1 \leq q < +\infty$, we define $\|u\|_{1,\,q,\,\mathcal{M}}$ the discrete $W^{1,q}$ semi-norm of u by:

$$\|u\|_{1,\,q,\,\mathcal{M}} := \Big(\int_\Omega |\boldsymbol{\nabla}_{\mathcal{M}} u|^q \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{q}}. \tag{C.6}$$

For all vector functions $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M}}(\Omega)$ and all $1 \leq q < +\infty$, we define $\|\boldsymbol{u}\|_{1,\,q,\,\mathcal{M}}$ the discrete $W^{1,q}$ semi-norm of $\boldsymbol{u}$ by:

$$\|\boldsymbol{u}\|_{1,\,q,\,\mathcal{M}} := \Big(\int_\Omega |\boldsymbol{\nabla}_{\mathcal{M}} \boldsymbol{u}|^q \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{q}}. \tag{C.7}$$

On the space $\mathrm{H}_{\mathcal{M},0}(\Omega)$, the semi-norm $\|\,.\,\|_{1,\,q,\,\mathcal{M}}$ is actually a norm. This is a consequence of the discrete Poincaré inequality (see Proposition C.10).

C.2 Some classical properties of the Crouzeix-Raviart finite elements

We first recall the following classical result.

Lemma C.1. *Let $\mathcal{M}$ be a triangulation of Ω. Let $\hat{K}$ be the reference element which is the d-simplex the vertices of which are $\hat{a}_0 = (0,..,0)$ and $\hat{a}_i = (0,..,1,..0)$ where all the components are zero except the i-th component which equals 1, for $i = 1,..,d$. For $K \in \mathcal{M}$, let $\mathcal{A}_K$ be the (unique) affine mapping which maps the vertices of $\hat{K}$ onto those of K, and let $\mathcal{B}_K$ be its jacobian matrix (which is constant over K).*

- *We have the following estimate on $\mathcal{B}_K$ and its inverse:*

$$\|\mathcal{B}_K\| := \sup_{\substack{\hat{\boldsymbol{x}} \in \mathbb{R}^d \\ |\hat{\boldsymbol{x}}|_{\mathbb{R}^d} = 1}} |\mathcal{B}_K \, \hat{\boldsymbol{x}}|_{\mathbb{R}^d} \leq \frac{h_K}{\hat{\varrho}}, \qquad \|\mathcal{B}_K^{-1}\| := \sup_{\substack{\boldsymbol{x} \in \mathbb{R}^d \\ |\boldsymbol{x}|_{\mathbb{R}^d} = 1}} |\mathcal{B}_K^{-1} \, \boldsymbol{x}|_{\mathbb{R}^d} \leq \frac{\hat{h}}{\varrho_K}, \tag{C.8}$$

  where $\hat{h}$ and $\hat{\varrho}$ only depend on d.

- *Let $K \in \mathcal{M}$. With a function u defined on K, we associate a function $\hat{u}$ defined on $\hat{K}$ by $\hat{u}(\hat{\boldsymbol{x}}) = u(\boldsymbol{x})$ where $\boldsymbol{x} = \mathcal{A}_K(\hat{\boldsymbol{x}})$. Let E be either the simplex K or one of its edges $\sigma \in \mathcal{E}(K)$. Then one has, for all $q \in [1, +\infty)$:*

$$\|u\|_{\mathrm{L}^q(E)} = \Big(\frac{|E|}{|\hat{E}|} \Big)^{\frac{1}{q}} \|\hat{u}\|_{\mathrm{L}^q(\hat{E})}, \tag{C.9}$$

$$\|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(E)} \leq \|\mathcal{B}_K^{-1}\| \Big(\frac{|E|}{|\hat{E}|} \Big)^{\frac{1}{q}} \|\boldsymbol{\nabla} \hat{u}\|_{\mathbf{L}^q(\hat{E})}, \tag{C.10}$$

$$\|\boldsymbol{\nabla} \hat{u}\|_{\mathbf{L}^q(\hat{E})} \leq \|\mathcal{B}_K\| \Big(\frac{|\hat{E}|}{|E|} \Big)^{\frac{1}{q}} \|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(E)}. \tag{C.11}$$

We now state a result which can be obtained by easy adaptations of [39, Sections 3 and 4].

Lemma C.2. *Let $\mathcal{M}$ be a triangulation of Ω. Let $K \in \mathcal{M}$ be a given simplex and σ be one of its edges. Then we have the following trace inequality: for all $q \in [1, +\infty)$:*

$$\|u\|_{\mathrm{L}^q(\sigma)} \leq \Big(d\,\frac{|\sigma|}{|K|}\Big)^{\frac{1}{q}} \big(\|u\|_{\mathrm{L}^q(K)} + h_K\,\|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(K)}\big), \qquad \forall u \in \mathrm{W}^{1,q}(K). \tag{C.12}$$

Moreover, we have the following local Poincaré-Wirtinger inequality. Denoting u_K the mean value of u over K:

$$\|u - u_K\|_{\mathrm{L}^q(K)} \leq C(d,q)\,h_K\,\|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(K)}, \qquad \forall u \in \mathrm{W}^{1,q}(K). \tag{C.13}$$

Let us now define the following interpolation operator from $\mathrm{H}^1_0(\Omega)$ onto $\mathrm{H}_{\mathcal{M},0}(\Omega)$:

$$I_{\mathcal{M}}: \left| \begin{array}{ccl} \mathrm{H}^1_0(\Omega) & \longrightarrow & \mathrm{H}_{\mathcal{M},0}(\Omega) \\ u & \longmapsto & I_{\mathcal{M}} u = \displaystyle\sum_{\sigma\in\mathcal{E}} \Big(\frac{1}{|\sigma|}\int_\sigma u \,\mathrm{d}\sigma(\boldsymbol{x})\Big)\,\zeta_\sigma. \end{array}\right. \tag{C.14}$$

Naturally, for a vector field $\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)$ $(= \mathrm{H}^1_0(\Omega)^d)$, $I_{\mathcal{M}}\boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$ is defined as follows:

$$I_{\mathcal{M}}\boldsymbol{u} = \sum_{\sigma\in\mathcal{E}} \Big(\frac{1}{|\sigma|}\int_\sigma \boldsymbol{u}\,\mathrm{d}\sigma(\boldsymbol{x})\Big)\,\zeta_\sigma.$$

Proposition C.3 (Properties of operator $I_{\mathcal{M}}$)**.** *Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_{\mathcal{M}} \leq \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (C.1)) for some positive constant θ_0. The operator $I_{\mathcal{M}}$ satisfies the following properties. For all $q \in [1, +\infty)$, there exists $C = C(\theta_0, q, d)$ such that:*

(i) Stability:

$$\|I_{\mathcal{M}} u\|_{1,\,q,\,\mathcal{M}} \leq C\;|u|_{\mathrm{W}^{1,q}(\Omega)}, \qquad \forall u \in \mathrm{W}^{1,q}_0(\Omega). \tag{C.15}$$

(ii) Approximation: For all $K \in \mathcal{M}$:

$$\begin{aligned} \|u - I_{\mathcal{M}} u\|_{\mathrm{L}^q(K)} + h_K\,\|\boldsymbol{\nabla}(u - I_{\mathcal{M}} u)\|_{\mathbf{L}^q(K)} \\ \leq C\,h_K^2\,|u|_{\mathbf{W}^{2,q}(K)}, \qquad \forall u \in \mathrm{W}^{2,q}(\Omega)\cap \mathrm{W}^{1,q}_0(\Omega). \end{aligned} \tag{C.16}$$

(iii) Preservation of the divergence:

$$\int_\Omega p\,\mathrm{div}_{\mathcal{M}}(I_{\mathcal{M}}\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = \int_\Omega p\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}, \qquad \forall p \in \mathrm{L}_{\mathcal{M}}(\Omega),\ \boldsymbol{u} \in \mathbf{W}^{1,q}_0(\Omega). \tag{C.17}$$

Remark C.1. *An operator which satisfies properties* (i) *and* (iii) *of the above proposition is called a* Fortin operator*. The interested reader is referred to [21] for a similar result in the case of the MAC scheme.*

Proof. We start with proving (C.15). Let $u \in \mathrm{W}_0^{1,q}(\Omega)$. We have $I_{\mathcal{M}} u = \sum_{\sigma\in\mathcal{E}} u_\sigma \zeta_\sigma$ where $u_\sigma = |\sigma|^{-1} \int_\sigma u \, \mathrm{d}\sigma(\boldsymbol{x})$. Since ζ_σ has it support in the neighboring cells K and L if $\sigma \in \mathcal{E}_{\mathrm{int}}$ with $\sigma = K|L$ and in K if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}(K)$, we have:

$$\begin{aligned}
\|I_{\mathcal{M}} u\|_{1,\,q,\,\mathcal{M}}^q &= \sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla}(I_{\mathcal{M}} u)|^q \, \mathrm{d}\boldsymbol{x} \\
&= \sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla}(I_{\mathcal{M}} u - u_K)|^q \, \mathrm{d}\boldsymbol{x} \\
&= \sum_{K\in\mathcal{M}} \int_K \Big|\boldsymbol{\nabla}\big(\sum_{\sigma\in\mathcal{E}(K)} (u_\sigma - u_K)\zeta_\sigma\big)\Big|^q \, \mathrm{d}\boldsymbol{x} \\
&\leq \sum_{K\in\mathcal{M}} \int_K \Big(\sum_{\sigma\in\mathcal{E}(K)} |u_\sigma - u_K| \, |\boldsymbol{\nabla}\zeta_\sigma|\Big)^q \, \mathrm{d}\boldsymbol{x}. \qquad \text{(C.18)}
\end{aligned}$$

Applying Hölder's inequality, then the trace inequality (C.12) and finally the Poincaré-Wirtinger inequality (C.13) we have:

$$\begin{aligned}
|u_\sigma - u_K| &\leq \frac{1}{|\sigma|^{\frac{1}{q}}} \|u - u_K\|_{\mathrm{L}^q(\sigma)} \\
&\leq \Big(\frac{d}{|K|}\Big)^{\frac{1}{q}} \big(\|u - u_K\|_{\mathrm{L}^q(K)} + h_K \, \|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(K)}\big) \\
&\leq C(q,d) \, h_K \Big(\frac{d}{|K|}\Big)^{\frac{1}{q}} \|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(K)}.
\end{aligned}$$

Injecting in (C.18) we get:

$$\begin{aligned}
\|I_{\mathcal{M}} u\|_{1,\,q,\,\mathcal{M}}^q &\leq C(q,d) \sum_{K\in\mathcal{M}} \frac{h_K^q}{|K|} \|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(K)}^q \int_K \Big(\sum_{\sigma\in\mathcal{E}(K)} |\boldsymbol{\nabla}\zeta_\sigma|\Big)^q \, \mathrm{d}\boldsymbol{x} \\
&\leq C'(q,d) \sum_{K\in\mathcal{M}} \frac{h_K^q}{|K|} \|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(K)}^q \sum_{\sigma\in\mathcal{E}(K)} \|\boldsymbol{\nabla}\zeta_\sigma\|_{\mathbf{L}^q(K)}^q.
\end{aligned}$$

Now applying (C.10) to ζ_σ for all $\sigma \in \mathcal{E}(K)$ and observing that $\widehat{\zeta_\sigma} = \zeta_{\hat\sigma}$ with $\|\boldsymbol{\nabla}\zeta_{\hat\sigma}\|_{\mathbf{L}^q(\hat K)}$ which only depends on d, we get:

$$\begin{aligned}
\|I_{\mathcal{M}} u\|_{1,\,q,\,\mathcal{M}}^q &\leq C''(q,d) \sum_{K\in\mathcal{M}} \frac{h_K^q}{\varrho_K^q} \|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(K)}^q \\
&\leq C''(q,d) \, \theta_0^q \sum_{K\in\mathcal{M}} \|\boldsymbol{\nabla} u\|_{\mathbf{L}^q(K)}^q \\
&= C''(q,d) \, \theta_0^q \, |u|_{\mathrm{W}^{1,q}(\Omega)}^q.
\end{aligned}$$

We now give the proof of (C.16). Let $u \in \mathrm{W}^{2,q}(\Omega) \cap \mathrm{W}_0^{1,q}(\Omega)$ and let $K \in \mathcal{M}$. Denote $I_{\hat K} \hat u \in P_1(\hat K)$ the polynomial

$$I_{\hat K} \hat u = \sum_{\hat\sigma\in\mathcal{E}(\hat K)} \hat u_{\hat\sigma} \zeta_{\hat\sigma}, \quad \text{where} \quad \hat u_{\hat\sigma} = |\hat\sigma|^{-1} \int_{\hat\sigma} \hat u \, \mathrm{d}\sigma(\hat{\boldsymbol{x}}).$$

Following the above lines, we can prove that for $m = 0, 1$, the operator $I_{\hat{K}}$ is continuous from $\mathrm{W}^{m+1,q}(\hat{K})$ onto $\mathrm{W}^{m,q}(\hat{K})$, with an operator norm which only depends on $\hat{K}$, q and m. Since $I_{\hat{K}} v = v$ for all polynomials $v \in P_1(\hat{K})$, the Bramble-Hilbert Lemma applies (see [4, Lemma 6]) and one has for all $0 \leq n \leq m$ with $m = 0, 1$:

$$|\hat{u} - I_{\hat{K}} \hat{u}|_{\mathrm{W}^{n,q}(\hat{K})} \leq C(\hat{K}, q, m)\, |\hat{u}|_{\mathrm{W}^{m+1,q}(\hat{K})}.$$

Performing the change of variables $\boldsymbol{x} = \mathcal{A}_K(\hat{\boldsymbol{x}})$ and observing that if I_K is the restriction of $I_{\mathcal{M}}$ to K, then one has $\widehat{I_K u} = I_{\hat{K}} \hat{u}$, we get:

$$|u - I_K u|_{\mathrm{W}^{n,q}(K)} \leq C(\theta_0, d, q, m, n)\, h_K^{m+1-n} |u|_{\mathrm{W}^{m+1,q}(K)},$$

which straightforwardly yields (C.16).

Finally, we prove (C.17). Let $p \in \mathrm{L}_{\mathcal{M}}(\Omega)$ et $\boldsymbol{u} \in \mathbf{H}_{\mathcal{M},0}(\Omega)$. Denote p_K the constant value of p on the simplex K.

$$\begin{aligned}
&\int_\Omega p \operatorname{div}_{\mathcal{M}}(I_{\mathcal{M}} \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = \sum_{K \in \mathcal{M}} \int_K p \operatorname{div}(I_{\mathcal{M}} \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = \sum_{K \in \mathcal{M}} p_K \int_K \operatorname{div}(I_{\mathcal{M}} \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} \\
&= \sum_{K \in \mathcal{M}} p_K \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma I_{\mathcal{M}} \boldsymbol{u} \cdot \boldsymbol{n}_{K,\sigma} \,\mathrm{d}\sigma(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} p_K \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma \sum_{\sigma' \in \mathcal{E}} \Big(\frac{1}{|\sigma'|} \int_{\sigma'} \boldsymbol{u} \,\mathrm{d}\sigma(\boldsymbol{x}) \Big) \zeta_{\sigma'} \cdot \boldsymbol{n}_{K,\sigma}.
\end{aligned}$$

We know that $\displaystyle\int_\sigma \zeta_{\sigma'} \,\mathrm{d}\sigma(\boldsymbol{x}) = |\sigma| \delta_\sigma^{\sigma'}$, which yields:

$$\int_\Omega p \operatorname{div}_{\mathcal{M}}(I_{\mathcal{M}} \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = \sum_{K \in \mathcal{M}} p_K \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma \boldsymbol{u} \cdot \boldsymbol{n}_{K,\sigma} \,\mathrm{d}\sigma(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} p_K \int_K \operatorname{div} \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} = \int_\Omega p \operatorname{div} \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}.$$

□

C.3 Discrete inf-sup property

We recall the following result (see [35]).

Lemma C.4. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^d$, $d \geq 1$. Then, there exists a linear operator $\mathcal{B}$ depending only on Ω with the following properties:*

(i) For all $q \in (1, +\infty)$,

$$\mathcal{B} : \mathrm{L}_0^q(\Omega) \to \mathbf{W}_0^{1,q}(\Omega).$$

(ii) For all $q \in (1, +\infty)$ and $p \in \mathrm{L}_0^q(\Omega)$,

$$\operatorname{div}(\mathcal{B}p) = p, \ \textit{a.e. in } \Omega.$$

(iii) For all $q \in (1, +\infty)$, there exists $C = C(q, \Omega)$, such that for any $p \in \mathrm{L}_0^q(\Omega)$:

$$|\mathcal{B}p|_{\mathbf{W}^{1,q}(\Omega)} \leq C\, \|p\|_{\mathrm{L}^q(\Omega)}.$$

We now prove that the pair of discrete spaces $(\mathrm{L}_{\mathcal{M},0}(\Omega), \mathbf{H}_{\mathcal{M},0}(\Omega))$ satisfies a discrete version of the above result. This is one of the main features of the Crouzeix-Raviart finite elements which is often referred to as the *discrete inf-sup property*.

Lemma C.5 (Discrete L^q inf-sup property)**.** *Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (3.1)) for some positive constant θ_0. Then, there exists a linear operator*

$$\mathcal{B}_{\mathcal{M}} : \mathrm{L}_{\mathcal{M},0}(\Omega) \longrightarrow \mathbf{H}_{\mathcal{M},0}(\Omega)$$

depending only on Ω and on the discretization such that the following properties hold:

(i) For all $p \in \mathrm{L}_{\mathcal{M},0}(\Omega)$,

$$\int_\Omega r \operatorname{div}_{\mathcal{M}}(\mathcal{B}_{\mathcal{M}} p) \,\mathrm{d}\boldsymbol{x} = \int_\Omega r\, p \,\mathrm{d}\boldsymbol{x}, \qquad \forall r \in \mathrm{L}_{\mathcal{M}}(\Omega).$$

(ii) For all $q \in (1, +\infty)$, there exists $C = C(q, d, \Omega, \theta_0)$, such that

$$\|\mathcal{B}_{\mathcal{M}} p\|_{1,q,\mathcal{M}} \le C \|p\|_{\mathrm{L}^q(\Omega)}.$$

Proof. Define $\mathcal{B}_{\mathcal{M}} = I_{\mathcal{M}} \circ \mathcal{B}$. For all $p \in \mathrm{L}_{\mathcal{M},0}(\Omega)$, we have by (C.17) :

$$\int_\Omega r \operatorname{div}_{\mathcal{M}}(\mathcal{B}_{\mathcal{M}} p) \,\mathrm{d}\boldsymbol{x} = \int_\Omega r \operatorname{div}_{\mathcal{M}}(I_{\mathcal{M}}(\mathcal{B} p)) \,\mathrm{d}\boldsymbol{x} = \int_\Omega r \operatorname{div}(\mathcal{B} p)) \,\mathrm{d}\boldsymbol{x} = \int_\Omega r\, p \,\mathrm{d}\boldsymbol{x}, \qquad \forall r \in \mathrm{L}_{\mathcal{M}}(\Omega).$$

Moreover, for $q \in (1, +\infty)$, we directly obtain $\|\mathcal{B}_{\mathcal{M}} p\|_{1,q,\mathcal{M}} \le C(\theta_0, q, d)\, |\mathcal{B} p|_{\mathrm{W}^{1,q}(\Omega)} \le C(\theta_0, q, d) \times C(q, \Omega)\, \|p\|_{\mathrm{L}^q(\Omega)}$, where $C(q, \Omega)$ is the constant given in Lemma C.4 and $C(\theta_0, q, d)$ the constant given in Proposition C.3. □

C.4 Discrete Sobolev inequalities

The aim of this section is to prove a discrete equivalent of the following Sobolev continuous injection result, the proof of which can be found for instance in [2].

Theorem C.6 (Sobolev, Gagliardo, Nirenberg)**.** *Let $1 \le p < \infty$.*

- *If $1 \le p < d$, then there exists a constant $C(p, d)$ such that:*

$$\|u\|_{\mathrm{L}^{p^\star}(\mathbb{R}^d)} \le C(p, d)\, \|\boldsymbol{\nabla} u\|_{\mathbf{L}^p(\mathbb{R}^d)}, \qquad \forall u \in \mathrm{W}^{1,p}(\mathbb{R}^d), \tag{C.19}$$

  where $p^\star = \dfrac{dp}{d-p}$. In particular, the injection $\mathrm{W}^{1,p}(\mathbb{R}^d) \subset \mathrm{L}^{p^\star}(\mathbb{R}^d)$ is continuous.

- *If $p \ge d$, then for all $q \in [p, \infty)$, there exists $C(p, q, d)$ such that:*

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^d)} \le C(p, q, d)\, \|\boldsymbol{\nabla} u\|_{\mathbf{L}^p(\mathbb{R}^d)}, \qquad \forall u \in \mathrm{W}^{1,p}(\mathbb{R}^d). \tag{C.20}$$

  In particular, the injection $\mathrm{W}^{1,p}(\mathbb{R}^d) \subset \mathrm{L}^q(\mathbb{R}^d)$ is continuous.

We first prove a technical result which will be useful in the following.

Lemma C.7. *Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (C.1)) for some positive constant θ_0. For a function v defined on K for all $K \in \mathcal{M}$ and for all $\sigma \in \mathcal{E}_{\rm int}$ with $\sigma = K|L$, we denote $[v]_\sigma =$ the jump of v across σ (* i.e.*$|[v]_\sigma(\boldsymbol{x})| = |v_{|K}(\boldsymbol{x}) - v_{|L}(\boldsymbol{x})|$, for all $\boldsymbol{x} \in \sigma$), and for $\sigma \in \mathcal{E}_{\rm ext}$, $\sigma \in K$, we denote $[v]_\sigma(\boldsymbol{x}) = v_{|K}(\boldsymbol{x})$, for all $\boldsymbol{x} \in \sigma$. Then, one has:*

- *For all $1 \le p < \infty$, there exists a constant $C = C(p, d, \theta_0)$ such that :*

$$\Big(\sum_{\sigma\in\mathcal{E}} \frac{1}{h_\sigma^{p-1}} \int_\sigma \big|[v]_\sigma\big|^p \,\mathrm{d}\sigma(\boldsymbol{x})\Big)^{\frac{1}{p}} \le C\, \|v\|_{1,\,p,\,\mathcal{M}}, \qquad \forall v \in \mathrm{H}_{\mathcal{M},0}(\Omega). \tag{C.21}$$

- *For all $1 < p < \infty$, there exists a constant $C = C(p, d, \theta_0)$ such that for all $\alpha > 1$:*

$$\Big(\sum_{\sigma\in\mathcal{E}} \int_\sigma \big|[|v|^\alpha]_\sigma\big| \,\mathrm{d}\sigma(\boldsymbol{x})\Big) \le \alpha\, C\, \|v\|^{\alpha-1}_{\mathrm{L}^{p'(\alpha-1)}(\Omega)}\, \|v\|_{1,\,p,\,\mathcal{M}}, \qquad \forall v \in \mathrm{H}_{\mathcal{M},0}(\Omega), \tag{C.22}$$

  where $1 < p' < \infty$ is given by $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. We first prove (C.21). Since for all $\sigma \in \mathcal{E}$, the integral of the jump of a function $v \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ across σ is zero, then by the mean value theorem, there exists $\boldsymbol{x}_\sigma \in \sigma$ such that $[v]_\sigma(\boldsymbol{x}_\sigma) = 0$. We then have:

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{E}} \frac{1}{h_\sigma^{p-1}} \int_\sigma \big|[v]_\sigma\big|^p \,\mathrm{d}\sigma(\boldsymbol{x}) \\
&= \sum_{\sigma\in\mathcal{E}} \frac{1}{h_\sigma^{p-1}} \int_\sigma \big|[v - v(\boldsymbol{x}_\sigma)]_\sigma\big|^p \,\mathrm{d}\sigma(\boldsymbol{x}) \\
&= \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} \frac{1}{h_\sigma^{p-1}} \int_\sigma \Big|\int_0^1 \Big(\boldsymbol{\nabla} v_{|K}(\boldsymbol{x}_\sigma + s(\boldsymbol{x}-\boldsymbol{x}_\sigma)) - \boldsymbol{\nabla} v_{|L}(\boldsymbol{x}_\sigma + s(\boldsymbol{x}-\boldsymbol{x}_\sigma))\Big)\cdot(\boldsymbol{x}-\boldsymbol{x}_\sigma)\,\mathrm{d}\boldsymbol{s}\Big|^p \,\mathrm{d}\sigma(\boldsymbol{x}) \\
&+ \sum_{\substack{\sigma\in\mathcal{E}_{\rm ext}\\ \sigma\in\mathcal{E}(K)}} \frac{1}{h_\sigma^{p-1}} \int_\sigma \Big|\int_0^1 \boldsymbol{\nabla} v_{|K}(\boldsymbol{x}_\sigma + s(\boldsymbol{x}-\boldsymbol{x}_\sigma))\cdot(\boldsymbol{x}-\boldsymbol{x}_\sigma)\,\mathrm{d}\boldsymbol{s}\Big|^p \,\mathrm{d}\sigma(\boldsymbol{x}).
\end{aligned}$$

In the following, we denote $\boldsymbol{\nabla} v(\boldsymbol{x}, s)$ instead of $\boldsymbol{\nabla} v(\boldsymbol{x}_\sigma + s(\boldsymbol{x}-\boldsymbol{x}_\sigma))$ to easy notation. The Cauchy-Schwarz inequality in $\mathbb{R}^d$ yields

$$\begin{aligned}
\sum_{\sigma\in\mathcal{E}} \frac{1}{h_\sigma^{p-1}} \int_\sigma \big|[v]_\sigma\big|^p \,\mathrm{d}\sigma(\boldsymbol{x}) \le \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} \frac{2^p}{h_\sigma^{p-1}} \int_\sigma |\boldsymbol{x}-\boldsymbol{x}_\sigma|^p \int_0^1 \Big(|\boldsymbol{\nabla} v_{|K}(\boldsymbol{x}, s)|^p + |\boldsymbol{\nabla} v_{|L}(\boldsymbol{x}, s)|^p\Big)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\sigma(\boldsymbol{x}) \\
+ \sum_{\substack{\sigma\in\mathcal{E}_{\rm ext}\\ \sigma\in\mathcal{E}(K)}} \frac{1}{h_\sigma^{p-1}} \int_\sigma |\boldsymbol{x}-\boldsymbol{x}_\sigma|^p \int_0^1 |\boldsymbol{\nabla} v_{|K}(\boldsymbol{x}, s)|^p \,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\sigma(\boldsymbol{x}).
\end{aligned}$$

Since $h_\sigma = \mathrm{diam}(\sigma)$, we have $|\boldsymbol{x}-\boldsymbol{x}_\sigma| \le h_\sigma$ for all σ. By Fubini's theorem, we obtain:

$$\sum_{\sigma\in\mathcal{E}} \frac{1}{h_\sigma^{p-1}} \int_\sigma \big|[v]_\sigma\big|^p \,\mathrm{d}\sigma(\boldsymbol{x}) \le C(p) \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} h_\sigma \int_\sigma |\boldsymbol{\nabla} v_{|K}|^p \,\mathrm{d}\sigma(\boldsymbol{x}). \tag{C.23}$$

By a finite dimension argument, there exists $C(d)$ such that for all polynomial function $\hat{w} \in P_1(\hat{K})$:

$$\sum_{\hat{\sigma}\in\mathcal{E}(\hat{K})} \int_{\hat{\sigma}} |\boldsymbol{\nabla}\hat{w}|^p \,\mathrm{d}\sigma(\hat{\boldsymbol{x}}) \leq C(d) \int_{\hat{K}} |\boldsymbol{\nabla}\hat{w}|^p \mathrm{d}\hat{\boldsymbol{x}}.$$

Applying (C.10) and (C.11) to $E = \sigma \in \mathcal{E}(K)$, we see that there exists $C(p, d, \theta_0)$ such that:

$$\sum_{\sigma\in\mathcal{E}(K)} h_\sigma \int_\sigma |\boldsymbol{\nabla}v_{|K}|^p \,\mathrm{d}\sigma(\boldsymbol{x}) \leq C(p, d, \theta_0) \int_K |\boldsymbol{\nabla}v_{|K}|^p \,\mathrm{d}\boldsymbol{x},$$

which combined with (C.23) proves (C.21).

Let us now prove (C.22). Since $[v]_\sigma(\boldsymbol{x}_\sigma) = 0$, we have $[|v|^\alpha]_\sigma(\boldsymbol{x}_\sigma) = 0$. In addition, since $\alpha > 1$, $|v|^\alpha$ is smooth where v is smooth and one has $\boldsymbol{\nabla}|v|^\alpha = \alpha \,\mathrm{sgn}(v)|v|^{\alpha-1}\boldsymbol{\nabla}v$. Following similar steps as those for the proof of (C.21), we find that:

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}} \int_\sigma \left|[|v|^\alpha]_\sigma\right| \mathrm{d}\sigma(\boldsymbol{x}) &\leq \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} h_\sigma \int_\sigma |\boldsymbol{\nabla}|v|^\alpha_{|K}| \,\mathrm{d}\sigma(\boldsymbol{x}) \\ &\leq \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} h_\sigma \,\alpha \int_\sigma |v_{|K}|^{\alpha-1} \,|\boldsymbol{\nabla}v_{|K}| \,\mathrm{d}\sigma(\boldsymbol{x}).\end{aligned}$$

Applying Hölder's inequality to the integral over σ, we obtain:

$$\sum_{\sigma\in\mathcal{E}} \int_\sigma \left|[|v|^\alpha]_\sigma\right| \mathrm{d}\sigma(\boldsymbol{x}) \leq \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} h_\sigma \,\alpha \|v\|^{\alpha-1}_{\mathrm{L}^{p'(\alpha-1)}(\sigma)} \,\|\boldsymbol{\nabla}v\|_{\mathbf{L}^p(\sigma)}.$$

Then, transporting this inequality to the reference element using (C.9) and (C.10), invoking a finite dimension argument on the reference element and finally transporting back to the simplex K thanks to (C.9) and (C.11), we prove that (with $\frac{1}{p} + \frac{1}{p'} = 1$) :

$$h_\sigma \,\alpha \|v\|^{\alpha-1}_{\mathrm{L}^{p'(\alpha-1)}(\sigma)} \,\|\boldsymbol{\nabla}v\|_{\mathbf{L}^p(\sigma)} \leq \alpha \,\frac{|\hat{K}|}{|\hat{\sigma}|} \,h_\sigma \,\frac{|\sigma|}{|K|} \,\|v\|^{\alpha-1}_{\mathrm{L}^{p'(\alpha-1)}(K)} \,\|\boldsymbol{\nabla}v\|_{\mathbf{L}^p(K)}.$$

The quantities $|\hat{K}|$ and $|\hat{\sigma}|$ only depend on d and by the regularity of the mesh, there exists $C(\theta_0)$ such that $h_\sigma \,\frac{|\sigma|}{|K|} \leq C(\theta_0)$ which concludes the proof of (C.22). □

We now want to prove a discrete equivalent to inequality (C.19) for discrete functions $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ considering the norm $\|u\|_{1,\,p,\,\mathcal{M}}$ instead of $\|\boldsymbol{\nabla}u\|_{\mathbf{L}^p(\mathbb{R}^d)}$. We begin with the case $p = 1$. Inequality (C.19) reads:

$$\|u\|_{\mathrm{L}^{1^\star}(\mathbb{R}^d)} \leq C(d)\|\boldsymbol{\nabla}u\|_{\mathbf{L}^1(\mathbb{R}^d)}, \quad \forall u \in \mathrm{W}^{1,1}(\mathbb{R}^d). \tag{C.24}$$

Let us first prove that this result extends to functions with bounded variations. For $u \in \mathrm{L}^1(\mathbb{R}^d)$, define:

$$\|u\|_{\mathrm{BV}} := \sup\Big\{\langle\boldsymbol{\nabla}u, \boldsymbol{\phi}\rangle_{\mathcal{D}',\mathcal{D}}, \text{ with } \boldsymbol{\phi} \in \mathcal{C}^\infty_c(\mathbb{R}^d)^d \text{ s.t. } \|\boldsymbol{\phi}\|_{\mathbf{L}^\infty(\mathbb{R}^d)} \leq 1\Big\}.$$

A function $u \in \mathrm{L}^1(\mathbb{R}^d)$ is said to have bounded variations if $\|u\|_{\mathrm{BV}} < \infty$. The space of such functions is denoted $\mathrm{BV}(\mathbb{R}^d)$. Let us prove that (C.24) holds true for functions in $\mathrm{BV}(\mathbb{R}^d)$ with $\|u\|_{\mathrm{BV}}$ instead of $\|\boldsymbol{\nabla} u\|_{\mathbf{L}^1(\mathbb{R}^d)}$. For all $u \in \mathrm{BV}(\mathbb{R}^d)$, there exists (see [1] for instance) a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{L}^1(\mathbb{R}^d)$ such that $u_n \to u$ in $\mathrm{L}^1(\mathbb{R}^d)$ and almost everywhere, and such that $\|\boldsymbol{\nabla} u_n\|_{\mathbf{L}^1(\mathbb{R}^d)} = \|u_n\|_{\mathrm{BV}} \to \|u\|_{\mathrm{BV}}$. The sequence $\big(\|u_n\|_{\mathrm{L}^{1^\star}(\mathbb{R}^d)}\big)_{n\in\mathbb{N}}$ is bounded and since $u_n \to u$ *a.e.*, Fatou's lemma gives $\|u\|_{\mathrm{L}^{1^\star}(\mathbb{R}^d)} \le \liminf_{n\to+\infty} \|u_n\|_{\mathrm{L}^{1^\star}(\mathbb{R}^d)}$. Letting $n \to +\infty$ in (C.24) written for u_n we obtain:

$$
\begin{aligned}
\|u\|_{\mathrm{L}^{1^\star}(\mathbb{R}^d)} &\le \liminf_{n\to+\infty} \|u_n\|_{\mathrm{L}^{1^\star}(\mathbb{R}^d)} \\
&\le C(d) \lim_{n\to+\infty} \|\boldsymbol{\nabla} u_n\|_{\mathbf{L}^1(\mathbb{R}^d)} \\
&= C(d) \lim_{n\to+\infty} \|u_n\|_{\mathrm{BV}} \\
&= C(d)\, \|u\|_{\mathrm{BV}}.
\end{aligned} \tag{C.25}
$$

Let us now prove that any function $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ can be extend to $\mathbb{R}^d$ to a function in $\mathrm{BV}(\mathbb{R}^d)$ with $\|u\|_{\mathrm{BV}} \lesssim \|u\|_{1,1,\mathcal{M}}$. This result, which will be satisfied under a regularity constraint on the mesh, is a consequence of Lemma C.7 which provides estimates on the jumps of the functions $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$.

Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_{\mathcal{M}} \le \theta_0$ (where $\theta_{\mathcal{M}}$ is defined by (C.1)) for some positive constant θ_0. Let $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$. We extend u by 0 outside Ω so that $u \in \mathrm{L}^1(\mathbb{R}^d)$. For all $\boldsymbol{\phi} \in \mathcal{C}_c^\infty(\mathbb{R}^d)^d$, we have:

$$
\begin{aligned}
\langle \boldsymbol{\nabla} u, \boldsymbol{\phi}\rangle_{\mathcal{D}',\mathcal{D}} = -\int_{\mathbb{R}^d} u \operatorname{div}\boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} = -\int_{\Omega} u \operatorname{div}\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} &= -\sum_{K\in\mathcal{M}} \int_K u \operatorname{div}\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} \\
&= \sum_{K\in\mathcal{M}} \Big(\int_K \boldsymbol{\nabla} u \cdot \boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} - \int_{\partial K} u\, \boldsymbol{\phi}\cdot \boldsymbol{n} \,\mathrm{d}\sigma(\boldsymbol{x})\Big) \\
&= \sum_{K\in\mathcal{M}} \Big(\int_K \boldsymbol{\nabla} u \cdot \boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} - \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma u\, \boldsymbol{\phi}\cdot \boldsymbol{n}_{K,\sigma} \,\mathrm{d}\sigma(\boldsymbol{x})\Big) \\
&= \sum_{K\in\mathcal{M}} \int_K \boldsymbol{\nabla} u \cdot \boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} - \sum_{\sigma\in\mathcal{E}} \int_\sigma [u]_\sigma\, \boldsymbol{\phi}\cdot \boldsymbol{n}_{K,\sigma} \,\mathrm{d}\sigma(\boldsymbol{x}) \\
&\le \Big(\sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla} u| \,\mathrm{d}\boldsymbol{x} + \sum_{\sigma\in\mathcal{E}} \int_\sigma |[u]_\sigma| \,\mathrm{d}\sigma(\boldsymbol{x})\Big) \|\boldsymbol{\phi}\|_{\mathbf{L}^\infty(\mathbb{R}^d)} \\
&\le C(d,\theta_0)\, \|u\|_{1,1,\mathcal{M}}\, \|\boldsymbol{\phi}\|_{\mathbf{L}^\infty(\mathbb{R}^d)}
\end{aligned} \tag{C.26}
$$

by definition of $\|u\|_{1,1,\mathcal{M}}$ and thanks to inequality (C.21) (with $p = 1$). By (C.26), we obtain that a function $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ (extend by 0 outside Ω) belongs to $\mathrm{BV}(\mathbb{R}^d)$ with :

$$
\|u\|_{\mathrm{BV}} \le C(d,\theta_0)\, \|u\|_{1,1,\mathcal{M}}. \tag{C.27}
$$

Combining (C.27) with (C.25), we therefore have proven the following result:

Proposition C.8 (Discrete continuous injection $W^{1,1} \subset L^{1^\star}$)**.** *Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_\mathcal{M} \le \theta_0$, with $\theta_0 > 0$. There exists a constant $C = C(d, \theta_0)$ such that:*

$$\|u\|_{\mathrm{L}^{1^\star}(\Omega)} \le C\, \|u\|_{1,\,1,\,\mathcal{M}}, \quad \forall u \in \mathrm{H}_{\mathcal{M},0}(\Omega). \tag{C.28}$$

We may now prove a discrete counterpart to the Sobolev inequality (C.19) for $1 \le p < d$:

Proposition C.9 (Discrete continuous injection $W^{1,p} \subset L^{p^\star}$ for $1 \le p < d$)**.** *Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_\mathcal{M} \le \theta_0$, with $\theta_0 > 0$. Assume that $1 \le p < d$. Then there exists a constant $C = C(p, d, \theta_0)$ such that:*

$$\|u\|_{\mathrm{L}^{p^\star}(\Omega)} \le C\, \|u\|_{1,\,p,\,\mathcal{M}}, \quad \forall u \in \mathrm{H}_{\mathcal{M},0}(\Omega), \tag{C.29}$$

where $p^\star = \dfrac{dp}{d-p}$. In addition, one has $C(p, d, \theta_0) \to +\infty$ as $p \to d$.

By interpolation of Lebesgue spaces, we have the following corollary, a consequence of which is a *discrete Poincaré inequality.*

Proposition C.10. *Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_\mathcal{M} \le \theta_0$, with $\theta_0 > 0$. Assume that $1 \le p < d$. Then, for all $q \in [p, p^\star]$, there exists a constant $C = C(p, q, d, \theta_0) > 0$ such that:*

$$\|u\|_{\mathrm{L}^{q}(\Omega)} \le C(p, q, d, \theta_0) \|u\|_{1,\,p,\,\mathcal{M}}, \quad \forall u \in \mathrm{H}_{\mathcal{M},0}(\Omega). \tag{C.30}$$

For $q = p$, this inequality is called the discrete Poincaré inequality*.*

Proof of Proposition (C.9). Let p such that $1 \le p < d$. Let $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ which we extend by 0 outside Ω. Let $\alpha > 1$. Since u is smooth on each simplex K, $|u|^\alpha$ is also smooth and one has $\boldsymbol{\nabla}|u|^\alpha = \alpha \operatorname{sgn}(u)|u|^{\alpha-1}\boldsymbol{\nabla} u$ on K. Performing the same calculation as in (C.26) with the function $|u|^\alpha$, we obtain, for all $\boldsymbol{\phi} \in \mathcal{C}_c^\infty(\mathbb{R}^d)^d$:

$$\begin{aligned}\langle \boldsymbol{\nabla}|u|^\alpha, \boldsymbol{\phi}\rangle_{\mathcal{D}',\mathcal{D}} &\le \Big(\sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla}|u|^\alpha| \,\mathrm{d}\boldsymbol{x} + \sum_{\sigma\in\mathcal{E}} \int_\sigma |[|u|^\alpha]_\sigma| \,\mathrm{d}\sigma(\boldsymbol{x})\Big) \|\boldsymbol{\phi}\|_{\mathbf{L}^\infty(\mathbb{R}^d)} \\ &\le \Big(\sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla}|u|^\alpha| \,\mathrm{d}\boldsymbol{x} + \alpha\, C(p, d, \theta_0)\, \|u\|^{\alpha-1}_{\mathrm{L}^{p'(\alpha-1)}(\Omega)}\, \|u\|_{1,\,p,\,\mathcal{M}}\Big) \|\boldsymbol{\phi}\|_{\mathbf{L}^\infty(\mathbb{R}^d)} \end{aligned} \tag{C.31}$$

thanks to inequality (C.22) of Lemma C.7. Moreover we have:

$$\sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla}|u|^\alpha| \,\mathrm{d}\boldsymbol{x} \le \alpha \sum_{K\in\mathcal{M}} \int_K |u|^{\alpha-1} |\boldsymbol{\nabla} u| \,\mathrm{d}\boldsymbol{x}.$$

Applying Hölder's inequality to the integral and then to the sum, we obtain (with $\frac{1}{p} + \frac{1}{p'} = 1$):

$$\begin{aligned}&\sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla}|u|^\alpha| \,\mathrm{d}\boldsymbol{x} \\ &\qquad \le \alpha \Big(\sum_{K\in\mathcal{M}} \int_K |u|^{p'(\alpha-1)} \,\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p'}} \Big(\sum_{K\in\mathcal{M}} \int_K |\boldsymbol{\nabla} u|^p \,\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}} = \alpha\, \|u\|^{\alpha-1}_{\mathrm{L}^{p'(\alpha-1)}(\Omega)}\, \|u\|_{1,\,p,\,\mathcal{M}}.\end{aligned}$$

We inject this inequality in (C.31) which yields that for all $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ and all $\alpha > 1$, we have $|u|^\alpha \in \mathrm{BV}(\mathbb{R}^d)$ avec $\||u|^\alpha\|_{\mathrm{BV}} \le \alpha\left(1 + C(p,d,\theta_0)\right) \|u\|^{\alpha-1}_{\mathrm{L}^{p'(\alpha-1)}} \|u\|_{1,\,p,\,\mathcal{M}}$. Inequality (C.25) applied to $|u|^\alpha$ then gives:

$$\|u\|^{\alpha}_{\mathrm{L}^{\alpha.1^\star}(\Omega)} = \||u|^\alpha\|_{\mathrm{L}^{1^\star}(\Omega)} \le \alpha\, C'(p,d,\theta_0)\|u\|^{\alpha-1}_{\mathrm{L}^{p'(\alpha-1)}(\Omega)} \|u\|_{1,\,p,\,\mathcal{M}}.$$

We then chose α such that $\alpha.1^\star = p'(\alpha-1)$ *i.e.*$\alpha = p'/(p'-1^\star)$ which gives (C.29) with $p^\star = \alpha.1^\star = p'1^\star/(p'-1^\star) = pd/(d-p)$. We may check that $\alpha > 1$ and $\alpha\, C'(p,d,\theta_0) \to \infty$ as $p \to d$ because $\alpha \to \infty$ as $p \to d$. □

Finally, for $p \ge d$, we have the following result.

Proposition C.11 (Discrete continuous injection $W^{1,p} \subset L^q$ for $p \ge d$)**.** *Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_{\mathcal{M}} \le \theta_0$, with $\theta_0 > 0$. Assume that $p \ge d$. Then, for all $q \in (p,\infty)$, there exists a constant $C = C(p,q,d,\theta_0) > 0$ such that:*

$$\|u\|_{\mathrm{L}^q(\Omega)} \le C\,\|u\|_{1,\,p,\,\mathcal{M}}, \quad \forall u \in \mathrm{H}_{\mathcal{M},0}(\Omega). \tag{C.32}$$

Proof. Let $q \in (p,\infty)$ and let $1 \le p_1 < d$ such that $p_1^\star = q$ (such a p_1 always exists because $p_1^\star \to \infty$ as $p_1 \to d$). Applying Proposition C.9, we obtain that for all $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$, $\|u\|_{\mathrm{L}^q(\Omega)} \le C(p_1,d,\theta_0)\|u\|_{1,\,p_1,\,\mathcal{M}}$. Then using Hölder's inequality, we have $\|u\|_{1,\,p_1,\,\mathcal{M}} \le C(\Omega,p,p_1)\,\|u\|_{1,\,p,\,\mathcal{M}}$. □

C.5 Compactness results

In this section, we prove a discrete counterpart to Rellich's compactness theorem. We obtain this result as a consequence of Kolmogorov's theorem which we recall (see [2] for a proof).

Theorem C.12 (Kolmogorov)**.** *Let Ω be an open bounded subset of $\mathbb{R}^d$, $d \ge 1$, $1 \le p < \infty$ and $A \subset \mathrm{L}^p(\Omega)$. Then A is relatively compact in $\mathrm{L}^p(\Omega)$ if, and only if, there exists an extension operator:*

$$\begin{array}{rccc} P: & A & \longrightarrow & \mathrm{L}^p(\mathbb{R}^d) \\ & u & \longmapsto & P(u) \end{array}$$

satisfying the following properties:

1. *$P(u) = u$ almost everywhere on Ω, for all $u \in A$,*
2. *The set $\{P(u),\, u \in A\}$ is bounded in $\mathrm{L}^p(\mathbb{R}^d)$,*
3. *$\|P(u)(.+\boldsymbol{y}) - P(u)\|_{\mathrm{L}^p(\mathbb{R}^d)} \to 0$ as $\boldsymbol{y} \to 0$, uniformly with respect to $u \in A$.*

C.5.1 Bounded sequence in the discrete $\mathrm{W}^{1,1}$ norm : compactness in L^1

We first establish an estimate on the translations of discrete functions $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$.

Proposition C.13 (Translations in L^1). *Let $\mathcal{M}$ be a triangulation of Ω such that $\theta_{\mathcal{M}} \le \theta_0$, with $\theta_0 > 0$. Then:*

$$\|u(.+\boldsymbol{y}) - u\|_{\mathrm{L}^1(\mathbb{R}^d)} \le |\boldsymbol{y}|\, C(d,\theta_0)\|u\|_{1,\,1,\,\mathcal{M}}, \quad \forall u \in \mathrm{H}_{\mathcal{M},0}(\Omega),\, \forall \boldsymbol{y} \in \mathbb{R}^d,$$

where $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ is extend to $\mathbb{R}^d$ setting $u = 0$ outside Ω, and $|\boldsymbol{y}|$ is the euclidian norm of $\boldsymbol{y} \in \mathbb{R}^d$.

Proof. Let $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. For all $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^d$, we have:

$$|u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x})| = \Big| \int_0^1 \boldsymbol{\nabla} u(\boldsymbol{x}+s\boldsymbol{y}) \cdot \boldsymbol{y}\, \mathrm{d}\boldsymbol{s} \Big| \le |\boldsymbol{y}| \int_0^1 |\boldsymbol{\nabla} u(\boldsymbol{x}+s\boldsymbol{y})|\, \mathrm{d}\boldsymbol{s}.$$

Integrating with respect to $\boldsymbol{x}$ and using Fubini's theorem, we obtain:

$$\|u(.+\boldsymbol{y}) - u\|_{\mathrm{L}^1(\mathbb{R}^d)} \le |\boldsymbol{y}| \int_{\mathbb{R}^d} |\boldsymbol{\nabla} u|\, \mathrm{d}\boldsymbol{x} = |\boldsymbol{y}|\, \|\boldsymbol{\nabla} u\|_{\mathbf{L}^1(\mathbb{R}^d)}. \tag{C.33}$$

By the density of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in $\mathrm{W}^{1,1}(\mathbb{R}^d)$, inequality (C.33) holds true for functions $u \in \mathrm{W}^{1,1}(\mathbb{R}^d)$. Then proceeding as in the proof of Proposition C.8, we prove that we can extend the result to functions $u \in \mathrm{BV}(\mathbb{R}^d)$:

$$\|u(.+\boldsymbol{y}) - u\|_{\mathrm{L}^1(\mathbb{R}^d)} \le |\boldsymbol{y}|\, \|u\|_{\mathrm{BV}}, \qquad \forall u \in \mathrm{BV}(\mathbb{R}^d).$$

Now let $u \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ which we extend by 0 outside Ω. According to (C.27), we have $u \in \mathrm{BV}(\mathbb{R}^d)$ with $\|u\|_{\mathrm{BV}} \le C(d,\theta_0)\, \|u\|_{1,\,1,\,\mathcal{M}}$ which yields the result. □

We deduce the following result which is a discrete counterpart of the compact injection of $\mathrm{W}^{1,1}(\Omega)$ in $\mathrm{L}^1(\Omega)$ for a bounded subset Ω.

Theorem C.14. *Let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be a sequence of regular triangulations of Ω* i.e. *a sequence satisfying $\theta_{\mathcal{M}_n} \le \theta_0$ for all $n \in \mathbb{N}$ with $\theta_0 > 0$. For all $n \in \mathbb{N}$, let $u_n \in \mathrm{H}_{\mathcal{M}_n,0}$ which we extend by 0 outside Ω. Assume that there exists $C \in \mathbb{R}$ such that $\|u_n\|_{1,1,\mathcal{M}_n} \le C$, $\forall\, n \in \mathbb{N}$. Then there exists a converging subsequence of $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^1(\mathbb{R}^d)$ and therefore in $\mathrm{L}^1(\Omega)$.*

Proof. We apply Kolmogorov's Theorem to the set $A = \cup_{n\in\mathbb{N}}\{u_n\}$. The operator P is the extension by 0 outside Ω. Hypothesis 1. is satisfied. Moreover Proposition C.8 shows that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{1^\star}(\Omega)$ and therefore in $\mathrm{L}^1(\Omega)$ and in $\mathrm{L}^1(\mathbb{R}^d)$ since $1^\star > 1$ and Ω is bounded. Therefore, hypothesis 2. is also satisfied. Finally, thanks to Proposition C.13 and to the fact that $\|u_n\|_{1,1,\mathcal{M}_n} \le C$, $\forall\, n \in \mathbb{N}$, we see that $\|u_n(.+\boldsymbol{y}) - u_n\|_{\mathrm{L}^p(\mathbb{R}^d)} \to 0$ as $\boldsymbol{y} \to 0$, uniformly with respect to $n \in \mathbb{N}$. Kolmogorov's Theorem therefore applies and gives the existence of a converging subsequence of $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^1(\mathbb{R}^d)$ and therefore in $\mathrm{L}^1(\Omega)$. □

C.5.2 Bounded sequence in the discrete $\mathrm{W}^{1,p}$ norm: compactness in L^p

Theorem C.15. *Let $1 \leq p < \infty$. Let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be a sequence of regular triangulations of Ω i.e. a sequence satisfying $\theta_{\mathcal{M}_n} \leq \theta_0$ for all $n \in \mathbb{N}$ with $\theta_0 > 0$. For all $n \in \mathbb{N}$, let $u_n \in \mathrm{H}_{\mathcal{M}_n,0}$ which we extend by 0 outside Ω. Assume that there exists $C \in \mathbb{R}$ such that $\|u_n\|_{1,p,\mathcal{M}_n} \leq C$, $\forall\, n \in \mathbb{N}$. Then there exists a converging subsequence of $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^p(\mathbb{R}^d)$ and therefore in $\mathrm{L}^p(\Omega)$.*

Proof. Since Ω is bounded, the fact that $\|u_n\|_{1,p,\mathcal{M}_n} \leq C$, $\forall\, n \in \mathbb{N}$ combined with Hölder's inequality, shows that the sequence $(\|u_n\|_{1,1,\mathcal{M}_n})_{n\in\mathbb{N}}$ is bounded. Thus, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{1^\star}(\Omega)$ and therefore in $\mathrm{L}^1(\Omega)$ and in $\mathrm{L}^1(\mathbb{R}^d)$. We can apply Theorem C.14, which yields the existence of a subsequence $(u_n)_{n\in\mathbb{N}}$, still denoted $(u_n)_{n\in\mathbb{N}}$, which converges in $\mathrm{L}^1(\Omega)$ towards some function $u \in \mathrm{L}^1(\Omega)$.

We conclude the proof as follows. Invoking Proposition C.9 or Proposition C.11, we get that $(u_n)_{n\in\mathbb{N}}$ is also bounded in $\mathrm{L}^q(\Omega)$ for some $q > p$. Upon extracting a new subsequence, we can assume that $u_n \rightharpoonup v$ with $v \in \mathrm{L}^q(\Omega)$. Since the distributional limit is unique, we have $u = v$ *i.e.* $u \in \mathrm{L}^q(\Omega)$. Interpolating L^p between L^1 and L^q, we get:

$$\|u_n - u\|_{\mathrm{L}^p(\Omega)} \leq \|u_n - u\|^{\beta}_{\mathrm{L}^1(\Omega)} \|u_n - u\|^{1-\beta}_{\mathrm{L}^q(\Omega)},$$

where $\beta \in [0,1]$ satisfies $\frac{1}{p} = \beta + \frac{1-\beta}{q}$. This proves that $u_n \to u$ strongly in $\mathrm{L}^p(\Omega)$. □

C.5.3 Regularity of the limit

At the continuous level, the strong limit in $\mathrm{L}^p(\Omega)$ of a sequence of functions which is bounded in $\mathrm{W}^{1,p}_0(\Omega)$ is actually in $\mathrm{W}^{1,p}_0(\Omega)$. We prove that this still holds true for converging sequences of discrete functions associated with a sequence of refined meshes.

Theorem C.16. *Let $1 \leq p < \infty$. Let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be a sequence of regular triangulations of Ω i.e. a sequence satisfying $\theta_{\mathcal{M}_n} \leq \theta_0$ for all $n \in \mathbb{N}$ with $\theta_0 > 0$. For all $n \in \mathbb{N}$, let $u_n \in \mathrm{H}_{\mathcal{M}_n,0}$ which we extend by 0 outside Ω. Assume that there exists $C \in \mathbb{R}$ such that $\|u_n\|_{1,p,\mathcal{M}_n} \leq C$, $\forall\, n \in \mathbb{N}$. Assume that $h_{\mathcal{M}_n} \to 0$ as $n \to +\infty$. Then:*

1. *There exists a subsequence of $(u_n)_{n\in\mathbb{N}}$, still denotes $(u_n)_{n\in\mathbb{N}}$, which converges in $\mathrm{L}^p(\Omega)$ towards some function $u \in \mathrm{L}^p(\Omega)$.*
2. *The limit u belongs to $\mathrm{W}^{1,p}_0(\Omega)$ with $\|\boldsymbol{\nabla} u\|_{\mathbf{L}^p(\Omega)} \leq C$.*
3. *The sequence $(\boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{u}_n)_{n\in\mathbb{N}}$ weakly converges to $\boldsymbol{\nabla u}$ in $\mathbf{L}^p(\Omega)$.*

Proof. The existence of a subsequence of $(u_n)_{n\in\mathbb{N}}$, still denoted $(u_n)_{n\in\mathbb{N}}$, which converges in $\mathrm{L}^p(\Omega)$ towards some $u \in \mathrm{L}^p(\Omega)$ is guaranteed by Theorem C.15. Extending u_n and u by 0 outside Ω, we have $u_n \to u$ in $\mathrm{L}^p(\mathbb{R}^d)$. Let us prove that $\boldsymbol{\nabla}_{\mathcal{M}_n} \boldsymbol{u}_n$ weakly converges to $\boldsymbol{\nabla u}$ in $\mathbf{L}^p(\mathbb{R}^d)$.

We have

$$\|\boldsymbol{\nabla}_{\mathcal{M}_n} u_n\|_{\mathbf{L}^p(\mathbb{R}^d)} = \|\boldsymbol{\nabla}_{\mathcal{M}_n} u_n\|_{\mathbf{L}^p(\Omega)} = \|u_n\|_{1,p,\mathcal{M}_n} \leq C.$$

Moreover for all $\boldsymbol{\phi} \in \mathcal{C}_c^\infty(\mathbb{R}^d)^d$:

$$\begin{aligned}\int_{\mathbb{R}^d} \boldsymbol{\nabla}_{\mathcal{M}_n} u_n \cdot \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} &= \sum_{K\in\mathcal{M}_n} \int_K \boldsymbol{\nabla} u_n \cdot \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} \\ &= \sum_{K\in\mathcal{M}_n} \Big(-\int_K u_n \operatorname{div} \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} + \int_{\partial K} u_n \,\boldsymbol{\phi} \cdot \boldsymbol{n} \,\mathrm{d}\sigma(\boldsymbol{x})\Big) \\ &= -\int_{\mathbb{R}^d} u_n \operatorname{div} \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} + R(u_n, \phi)\end{aligned}$$

with

$$R(u_n, \phi) = \sum_{\sigma\in\mathcal{E}_n} \int_\sigma [u_n]_\sigma \,\boldsymbol{\phi} \cdot \boldsymbol{n}_{K,\sigma} \,\mathrm{d}\sigma(\boldsymbol{x}) = \sum_{\sigma\in\mathcal{E}_n} \int_\sigma [u_n]_\sigma \,(\boldsymbol{\phi} - \boldsymbol{\phi}(\boldsymbol{x}_\sigma)) \cdot \boldsymbol{n}_{K,\sigma} \,\mathrm{d}\sigma(\boldsymbol{x}),$$

since the integral of the jump of $u_n \in \mathrm{H}_{\mathcal{M},0}(\Omega)$ across σ is zero. Applying Hölder's inequality, we see that for all $1 < p < \infty$:

$$|R(u_n, \phi)| \leq \Big(\sum_{\sigma\in\mathcal{E}_n} \frac{1}{h_\sigma^{p-1}} \int_\sigma \big|[u_n]_\sigma\big|^p \,\mathrm{d}\sigma(\boldsymbol{x})\Big)^{\frac{1}{p}} \Big(\sum_{\sigma\in\mathcal{E}_n} \int_\sigma h_\sigma^{\frac{p'(p-1)}{p}} |\boldsymbol{\phi} - \boldsymbol{\phi}(\boldsymbol{x}_\sigma)|^{p'} \,\mathrm{d}\sigma(\boldsymbol{x})\Big)^{\frac{1}{p'}}.$$

Observing that $p' = p/(p-1)$ and using inequality (C.21) we get:

$$|R(u_n, \phi)| \leq \; C(p, d, \theta_0) \,\|u_n\|_{1,p,\mathcal{M}_n} \Big(\sum_{\sigma\in\mathcal{E}_n} \int_\sigma h_\sigma |\boldsymbol{\phi} - \boldsymbol{\phi}(\boldsymbol{x}_\sigma)|^{p'} \,\mathrm{d}\sigma(\boldsymbol{x})\Big)^{\frac{1}{p'}}.$$

Since $\boldsymbol{\phi}$ is smooth, we have $|\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{\phi}(\boldsymbol{x}_\sigma)| \leq h_{\mathcal{M}_n} \|\boldsymbol{\nabla}\boldsymbol{\phi}\|_{\mathbf{L}^\infty(\mathbb{R}^d)}$ for all $\boldsymbol{x} \in \sigma$. Hence, for all $1 < p < \infty$:

$$|R(u_n, \phi)| \leq C\, h_{\mathcal{M}_n},$$

and a similar results holds for $p = 1$. Since $u_n \to u$ strongly in $\mathrm{L}^p(\mathbb{R}^d)$, we obtain that for all $\boldsymbol{\phi} \in \mathcal{C}_c^\infty(\mathbb{R}^d)^d$:

$$\int_{\mathbb{R}^d} \boldsymbol{\nabla}_{\mathcal{M}_n} u \cdot \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} \to -\int_{\mathbb{R}^d} u \operatorname{div} \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x}, \quad \text{as } n \to +\infty,$$

and by density, $\boldsymbol{\nabla}_{\mathcal{M}_n} u_n \to \boldsymbol{\nabla} u$ weakly in $\mathbf{L}^p(\mathbb{R}^d)$ thus in $\mathbf{L}^p(\Omega)$. Since for all $n \in \mathbb{N}$,

$$\|\boldsymbol{\nabla}_{\mathcal{M}_n} u_n\|_{\mathbf{L}^p(\mathbb{R}^d)} = \|u_n\|_{1,p,\mathcal{M}_n} \leq C,$$

we deduce that $\boldsymbol{\nabla} u \in \mathrm{L}^p(\mathbb{R}^d)$ with $\|\boldsymbol{\nabla} u\|_{\mathbf{L}^p(\mathbb{R}^d)} = \|\boldsymbol{\nabla} u\|_{\mathbf{L}^p(\Omega)} \leq C$. Since $u = 0$ outside Ω, this implies that $u \in \mathrm{W}_0^{1,p}(\Omega)$ with $\|\boldsymbol{\nabla} u\|_{\mathbf{L}^p(\Omega)} \leq C$. □

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